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MATHÉMATIQUE

La quasi-solution spéciale du système parabolique d'équations

par

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Présenté par A. MOSTOWSKI le 29 mai, 1961

Introduction

Dans une note récente [1] nous avons rappelé la définition de la solution de S. D. Eidelman [2] d'un système parabolique d'équations aux coefficients constants. La dimension n de l'espace euclidien E_n, contenant l'ensemble Ω, où le système fut défini, ne dépasse pas M, l'ordre de ce système ($n \leqslant M$). Cette solution fut nommée la *quasi-solution spéciale* (voir la définition (1,1) de la note [1]) et fut appliquée à la méthode de W. Pogorzelski [3] pour construire la solution fondamentale du cas général ($n \leqslant M$).

Dans le travail que voici cette quasi-solution spéciale, si utile dans ses applications, sera étudiée en détail.

Nous adoptons toutes les définitions de l'article [1].

1. Le système d'équations à étudier est le suivant:

$$(1,1) \qquad \pi^{(a)}(u) - \partial u_a/\partial t \equiv \sum_{1 \leqslant j \leqslant N}^{(M)} A_{aj}^{(M)}(Z, \zeta)\, D^M(u_j) - \partial u_a/\partial t \equiv 0, \qquad a = 1, \ldots, N,$$

la somme (1,1) étant étendue sur toutes les suites (M), [1], $Z \in \Omega$, $\zeta > 0$.

Les hypothèses sur les coefficients du système (1,1) sont données ci-dessous:

HYPOTHÈSE I. *Les coefficients* $A_{aj}^{(M)}(Z, \zeta)$ *sont holdériens, donc*:

$$(2,1) \qquad |A_{aj}^{(M)}(Z, \zeta) - A_{aj}^{(M)}(Z', \zeta')| \leqslant C[|ZZ'|^h + |\zeta - \zeta'|^{h'}],$$

$$Z, Z' \in \bar{\Omega}' = \Omega'; \qquad \zeta, \zeta' \geqslant 0, \qquad 0 < h \leqslant 1; \qquad 0 < h' \leqslant 1; \qquad h_1 = \min(h, Mh').$$

HYPOTHÈSE II. *Le système* (1,1) *est parabolique selon Petrovsky*, [4], (voir aussi [3], formule (5) et (7)].

Rappelons que $[W^{Z,\zeta}]$ et $[{}_AW^{Z,\zeta}]$, $A = (a_1, \ldots, a_n)$, $|A| > 0$, désignent respectivement les matrices des solutions du système (1,1) — des quasi-solutions de W. Pogorzelski (voir [3], par. 2) — et des quasi-solutions spéciales (voir [1], définition (1,1) et formules (3,1) à (6,1)).

Rappelons aussi que l'hypothèse II permet d'affirmer que la matrice $[W^{Z,\zeta}]$ est formée des fonctions de la classe Z_q^q, $q = M/M-1$, d'après Gelfand et Chiloff (voir [5] et [3], par. 3). Il en résulte l'inégalité bien connue (voir [3], (33)):

$$(3,1) \qquad |D_X^k[W^{Z,\zeta}]| \leqslant C_k (t-\tau)^{-(n+k)/M} \exp\{-c_k [|XY|/(t-\tau)^{1/M}]^q\}.$$

C_k et c_k étant des constantes positives et $(k) = (k_1, ..., k_n)$, $k = k_1 + ... + k_n$. Il résulte de la définition de la quasi-solution spéciale (voir [1] par .1) que:

$$(4,1) \qquad {}_AW_{\alpha\beta}^{Z,\zeta} = \frac{1}{(M-n+1)!} [\xi_1 \partial/\partial x_1 + \cdots + \xi_n \partial/\partial x_n]^{(M-n+1)} W_{\alpha\beta}^{Z,\zeta}|_{X-Y=A'},$$

où:

$$A' = A + \eta(X - Y - A), \qquad \eta = (\eta_1, ..., \eta_n); \qquad |\eta| < 1; \qquad \xi_i = x_i - y_i - a_i, \quad i = 1, ..., n.$$

On a ainsi (voir [2] par. 4, formule (1)) pour $0 \leqslant k \leqslant M-n$:

$$(5.1) \qquad D_X^k({}_AW_{\alpha\beta}^{Z,\zeta}) = D_X^k(W_{\alpha\beta}^{Z,\zeta}) - \sum_{p=1}^{M-n-k} \frac{1}{p!} [\xi_1 \partial/\partial x_1 + \cdots + \xi_n \partial/\partial x_n]^p D_X^k(W_{\alpha\beta}^{Z,\zeta}) =$$

$$= \frac{1}{l!} [\xi_1 \partial/\partial x_1 + \cdots + \xi_n \partial/\partial x_n]^{(l)} D_X^k [W_{\alpha\beta}^{Z,\zeta}]|_{X-Y=A'}, \qquad l = M-n-k+1.$$

De la définition (1,1), [1] et de la formule (5,1) résultent les inégalités suivantes:

$$(6,1) \qquad |D_X^k[{}_AW_{\alpha\beta}^{Z,\zeta}]| \leqslant C'_k (t-\tau)^{-(1+1/M)} \exp\{-c'_k [|A'|/(t-\tau)^{1/M}]^q\}, \quad 0 \leqslant k \leqslant M-n$$

où

$$C'_k = C_{l+k} n^l \max_{X,\, Y \in \Omega'} |X - Y - A|^l / l!; \qquad c'_k = c_{l+k},$$

$$(7,1) \qquad D_X^k[{}_AP_{\alpha\beta}^{Z,\zeta}] \equiv 0, \qquad \text{si} \qquad k \geqslant M-n+1,$$

donc

$$(8,1) \qquad D_X^k[{}_AW_{\alpha\beta}^{Z,\zeta}] \equiv D_X^k[W_{\alpha\beta}^{Z,\zeta}], \qquad k \geqslant M-n+1.$$

2. La définition (1,1), [1] et les formules (6,1) à (8,1) donnent l'évaluation

$$(1,2) \qquad |D_X^k[{}_AW_{\alpha\beta}^{Z,\zeta}]| \leqslant \begin{cases} \mathcal{C}^{te}/(t-\tau)^{1+1/M}, & t-\tau > 1, \quad k = 0, 1, ...; X, Y, Z \in \Omega' \\ \mathcal{C}^{te}\,(t-\tau)^{-\mu}|XY|^{-(n+k-\mu M)}, & 1-1/M < \mu < 1, 0 < t-\tau \leqslant 1 \end{cases}$$

pour $k = 0, 1, ...$.

On démontrera maintenant le

LEMME (1,2). *Les hypothèses I et II étant admises, la quasi-solution spéciale (définition* (1,1), [1]) *et ses dérivées sont lipschitiennes par rapport à l'acroissement de la variable spatiale X, à savoir qu'on a pour* $k = 0, 1, ...$ *les inégalités*:

$$(2,2) \qquad |D_X^k[{}_AW_{\alpha\beta}^{Z,\zeta}(X,t;Y,\tau) - {}_AW_{\alpha\beta}^{Z,\zeta}(X',t;Y,\tau)]| \leqslant$$

$$\leqslant \begin{cases} \mathcal{C}^{te}|XX'|/(t-\tau)^{1+1/M}, & t-\tau > 1 \\ \mathcal{C}^{te}|XX'|(t-\tau)^{-\mu}|XY|^{-(n+k+1-\mu M)}, & 0 < t-\tau \leqslant 1. \end{cases}$$

Il résulte de la formule (8,1) que pour $k > M-n$ l'évaluation (2,2) n'est que l'évaluation de l'accroissement de la quasi-solution donnée dans [3]. Il suffit donc

de démontrer (2,2) pour $k \leqslant M - n$. Dans ce but on a recours au théorème de la moyenne et à l'évaluation (1,2), où l'indice k doit être remplacé par $k+1$. Le Lemme (1,2) se trouve donc démontré.

Pour abréger posons:

$$|XY|_A = \min(|XY|, A), \tag{3,2}$$

$$L = \sup_{k_1+\ldots+k_n=M} |A_{\alpha\beta}^{(M)}(Z,\zeta) - A_{\alpha\beta}^{(M)}(Z',\zeta')| \tag{4,2}$$

et rappelons la formule (59) de l'article [3]:

$$|D_X^k [W_{\alpha\beta}^{Z,\zeta} - W_{\alpha\beta}^{Z',\zeta'}]| \leqslant C_k'(t-\tau)^{-(n+k)/M} \exp\{-c_k' [|XY|/(t-\tau)^{1/M}]^q\} L. \tag{5,2}$$

Nous déduisons de cette formule et de la définition (1,1), [1] le

LEMME (2,2). *Les hypothèses I et II étant admises, la quasi-solution spéciale et ses dérivées sont soumises à l'inégalité suivante*:

$$|D_X^k [{}_A W_{\alpha\beta}^{Z,\zeta} - {}_A W_{\alpha\beta}^{Z',\zeta'}]| \leqslant \begin{cases} \mathcal{C}^{te}(t-\tau)^{-(1+1/M)} L; & t-\tau > 1 \\ \mathcal{C}^{te}(t-\tau)^{-\mu} |XY|_A^{(n+k-\mu M)} L; & 0 < t-\tau \leqslant 1. \end{cases} \tag{6,2}$$

Démonstration. 1) $k+n \geqslant M+1$. Dans ce cas on a l'inégalité (8,1) qui donne, conjointement avec les formules (5,2) et (4,2), l'inégalité (6,2).

2) $k+n < M+1$. Dans ce cas on déduit des formules (4,1) et (5,1) l'égalité suivante:

$$D_X^k({}_A W_{\alpha\beta}^{Z,\zeta} - {}_A W_{\alpha\beta}^{Z',\zeta'}) \equiv \frac{1}{l!}\left[\xi_1 \frac{\partial}{\partial x_1} + \ldots + \xi_n \frac{\partial}{\partial x_n}\right]^{(l)} D_X^k (W_{\alpha\beta}^{Z,\zeta} - W_{\alpha\beta}^{Z',\zeta'})|_{X-Y=A'}, \tag{7,2}$$

où

$$l = M - n - k + 1 \tag{8,2}$$

et A' est à trouver dans la formule (4,1), η dépendant aussi bien de (Z, ζ) que de (Z', ζ').

Observons que le domaine Ω' est supposé *borné*. De là et des formules (8,2), (7,2) et (5,2) avec la définition (4,2) résulte la première des inégalités (6,2).

Ecrivons, en partant des formules (5,1) et (5,2), ce qui suit:

$$|D_X^k({}_A W_{\alpha\beta}^{Z,\zeta} - {}_A W_{\alpha\beta}^{Z',\zeta'})| \leqslant [C_k'(t-\tau)^{-(n+k)/M} \exp\{-c_k' [|XY|/(t-\tau)^{1/M}]^q\} + \tag{9,2}$$

$$+ \sum_{p=0}^{M-n-k} \operatorname*{Max}_{X,Y\in\Omega'} |X-Y-A|^p n^p/p!\, C_{k+p}(t-\tau)^{-(n+k+p)/M} \times$$

$$\times \exp\{-c_{k+p}' [|A|/(t-\tau)^{1/M}]^q\}] \cdot L,$$

d'où résulte facilement la dernière formule (6,2). Le Lemme (2,2) se trouve ainsi démontré.

3. THÉORÈME (1,3). *Les hypothèses I et II étant admises, la quasi-solution spéciale* $[{}_A W^{Z,\zeta}]$ *est la solution du système*:

$$\pi^{(\alpha)}[u] - \partial u_\alpha/\partial t \equiv \partial({}_A P_{\alpha\beta}^{Z,\zeta})/\partial t, \quad \alpha, \beta = 1, \ldots, N; \quad \beta = \text{const.} \tag{1,3}$$

Cette quasi-solution serait alors égale à la quasi-solution $W^{Z,\zeta}_{\alpha\beta}$ *moins le potentiel du système* (1,3) *et de la charge spatiale* $\partial\,[{}_AP^{Z,\zeta}_{\alpha\beta}]/\partial t$. *On aurait donc:*

$$(2,3)\quad {}_AW^{Z,\zeta}_{\alpha\beta} = W^{Z,\zeta}_{\alpha\beta} - \int_\tau^t \int_{E_n} \sum_{\gamma=1}^{N} W^{Z,\zeta}_{\alpha\eta}(X,t;\pi,\theta)\,\partial\,[{}_AP^{Z,\zeta}_{\gamma\beta}(\pi,\theta;Y,\tau)]/\partial\theta\,d\pi\,d\theta = W^{Z,\zeta}_{\alpha\beta} - V^{Z,\zeta}_{\alpha\beta}, \qquad Z\in\Omega',\ \zeta>0.$$

Démonstration. Nous avons pour $k = M$ l'identité (8,1), alors:

$$(3,3)\quad \pi^{(\alpha)}\,[{}_AW^{Z,\zeta}_{\alpha\beta}] = \pi^{(\alpha)}\,[W^{Z,\zeta}_{\alpha\beta}].$$

Par définition, [3], on a:

$$(4,3)\quad \pi^{(\alpha)}(W^{Z,\zeta}_{\alpha\beta}) - \partial\,(W^{Z,\zeta}_{\alpha\beta})/\partial t \equiv 0, \qquad \alpha,\beta = 1,\ldots,N,\ \beta = \text{const}$$

et on a la définition (1,1), [1], ce qui donne l'identité (1,3) pour $u_\alpha = {}_AW^{Z,\zeta}_{\alpha\beta}$. Donc la formule (1,3) est démontrée.

Observons maintenant qu'on peut écrire, d'après les définitions (1,1), [1], (4,3) et la formule (5,1) pour $k = 0$:

$$(5,3)\quad \partial\,({}_AP^{Z,\zeta}_{\alpha\beta})/\partial t = \sum_{1\leqslant j\leqslant N}^{(M)} A^{(M)}_{\alpha j}(Z,\zeta) \sum_{p=0}^{M-n} \frac{1}{p!} \times \times (\xi_1\,\partial/\partial x_1 + \cdots + \xi_n\,\partial/\partial x_n)^{(p)}\,D^M\,[W^{Z,\zeta}_{j\beta}(X,t;Y.\tau)]|_{X-Y=A}$$

avec $|A| \neq 0$.

La fonction (5,3) est donc un polynôme de la variable spatiale X; elle est bornée et intégrable pour $X\in\Omega'$; $0 \leqslant \tau \leqslant t$. Dans E_n elle est limitée exponentiellement pour chaque M. Dès lors on peut appliquer à l'intégrale (2,3) le théorème 6 de l'article [6]. Il s'agit d'un cas particulier, car chez nous les coefficients de l'opération $\pi^{(\alpha)}$ sont, comme fonctions de (Z,ζ), constants par rapport à (X,t), (Y,τ), (Π,θ), et la densité de la charge spatiale (2,3) est bien holdérienne par rapport à la variable de l'intégration π. Nous pouvons donc écrire en tenant compte de la formule (2,3):

$$(6,3)\quad \pi^{(\alpha)}(V^{Z,\zeta}_{\alpha\beta}) - \partial\,(V^{Z,\zeta}_{\alpha\beta})/\partial t = \int_\tau^t \int_{E_n} \sum_{\gamma=1}^{N} [\pi^{(\alpha)}(W^{Z,\zeta}_{\alpha\gamma}) - \partial - (W^{Z,\zeta}_{\alpha\gamma})/\partial t] \times \times \partial({}_AP^{Z,\zeta}_{\gamma\beta})/\partial\theta\,d\Pi d\theta - \partial\,[P^{Z,\zeta}_{\alpha\beta}(X,t;Y,\tau)]/\partial t.$$

Les formules (4,3) et (6,3) démontrent que la charge spatiale $-V^{Z,\zeta}_{\alpha\beta}$ est la solution du système (1,3).

La fonction $-V^{Z,\zeta}_{\alpha\beta}$, (Y,τ) étant constante, converge vers zéro avec $t\to\tau+0$; il en est de même pour le polynôme $-{}_AP^{Z,\zeta}_{\alpha\beta}$. Les deux fonctions sont les solutions du système (1,3) dans $(E_n,(\tau,T)$. Elles sont limitées exponentiellement pour chaque M entier. Les coefficients du système (1,3) sont constants. Dans ces conditions il y a des théorèmes d'unicité bien connus (voir p.ex. [6] ou [7]) *). De sorte, que nous pouvons écrire:

$$(7,3)\quad {}_AP^{Z,\zeta}_{\alpha\beta} \equiv V^{Z,\zeta}_{\alpha\beta}.$$

*) Les théorèmes d'unicités I ou II de l'auteur de l'article actuel ne s'appliquent pas, les solutions ni „les parties droites" des équations n'étant pas bornées dans $[E_n, (O,T)]$.

Cette formule et la définition (1,1), [1], démontrent la formule (2,3), et de ce fait achèvent la démonstration du Théorème (1,3).

Il nous faut encore définir et étudier l'intégrale généralisée de Poisson- Weierstrass de la quasi-solution spéciale ${}_A W^{Z,\zeta}_{\alpha\beta}(X, t;\ Y, \tau)$. Soit une matrice $[\varrho(Y, \tau)]$ des fonctions bornées et intégrables dans $(\Omega', \langle 0, T\rangle)$, continues au point (X, t). Posons:

$$ {}_A J_{\alpha\beta}(X, t, \tau) \equiv \int_{\Omega'} \sum_{\gamma=1}^{N} {}_A W^{Y,\tau}_{\alpha\gamma}(X, t;\ Y, \tau)\, \varrho_{\gamma\beta}(Y, \tau)\, dY. $$

Il résulte des inégalités (1,2) pour $k = 0$ que cette intégrale existe. Il résulte du théorème 1, [3], et de la définition (1,1), [1], que:

$$ \lim_{t \to \tau + 0} {}_A J_{\alpha\beta}(X, t, \tau) = \varrho_{\alpha\beta}(X, t) \tag{8,3} $$

car $|A| > 0$ et alors:

$$ \int_{\Omega'} \sum_{\gamma=1}^{N} {}_A P^{Y,\tau}_{\alpha\gamma}(X, t;\ Y, \tau)\, \varrho_{\gamma\beta}(Y, \tau)\, dY \to 0, \qquad t \to \tau + 0. \tag{9,3} $$

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OUVRAGES CITÉS

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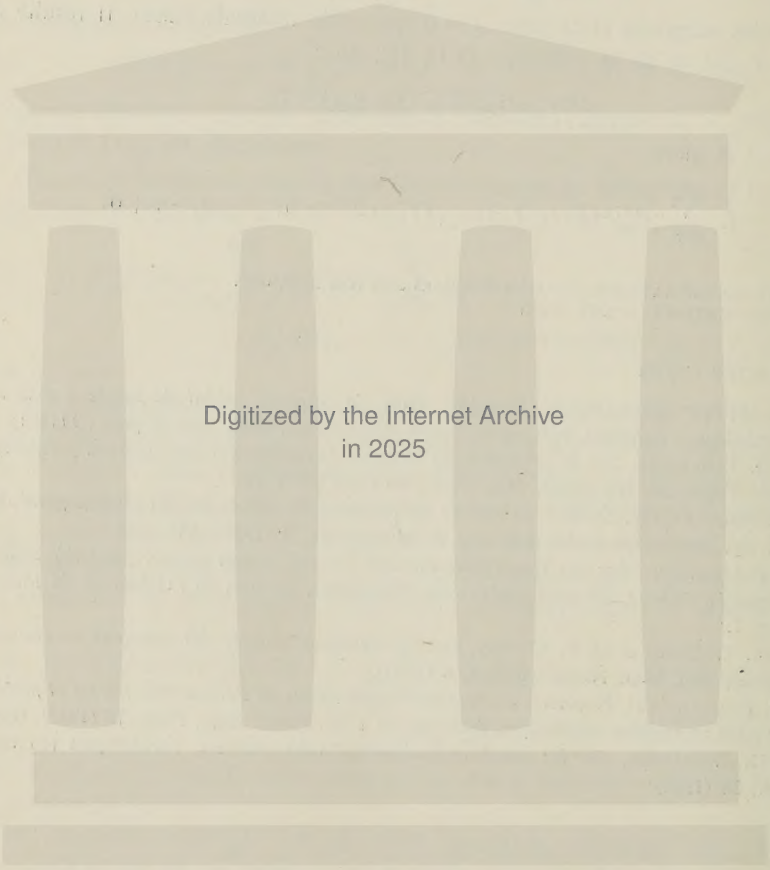

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MATHEMATICS

Measurable Cardinals and Constructible Sets

by

DANA SCOTT

Presented by A. MOSTOWSKI on April 28, 1961

A cardinal number $\mathfrak{m}$ will be called measurable if and only if there is a set X of cardinality $\mathfrak{m}$ and a non-trivial, real-valued, countably additive measure μ defined on all subsets of X. (The term non-trivial can be taken to mean that $\mu(X) = 1$ and $\mu(\{x\}) = 0$, for all $x \in X$). If $2^{\aleph_0} = \aleph_1$, Banach and Kuratowski [1] proved that $\aleph_1$ is not measurable. Ulam [12] proved that if there is a measurable cardinal, then either $2^{\aleph_0}$ is measurable or there exists a 2-valued measurable cardinal (2-valued in the sense that the measure μ can be assumed to take on only the values 0 and 1). Ulam and Tarski showed that no cardinal less than the first strongly inaccessible cardinal beyond $\aleph_0$ can be 2-valued measurable (cf. [12], esp. footnote 1, p. 146). Last year, using some new results of Hanf, Tarski proved [11] that many inaccessibles, in particular the first beyond $\aleph_0$, are not 2-valued measurable (for other proofs cf. [6] and [2]). Even though the least 2-valued measurable cardinal, if it exists at all, now appears to be incredibly large since Tarski's results apply to a seemingly inexhaustible number of inaccessible cardinals, it still seems plausible to many people including the author to assume that such cardinals do exist. However, this assumption has some surprising consequences, for, as shall be outlined below, we can show that the existence of measurable cardinals contradicts Gödel's axiom of constructibility.

We shall work within the system of [4] but shall not follow the notation of [4] too closely. The axiom $V = L$ is assumed in the form of the following statement:

(*) *If M is a class such that*

(i) $M \subseteq \boldsymbol{P}M \subseteq \bigcup_{x \in M} \boldsymbol{P}x$;

(ii) $x - y$, $\bigcup x$, $\breve{x}$, $x \restriction x$, $E \restriction x \in M$, *for all* $x, y \in M$;

then $V = M$.

(Above, the symbol $\boldsymbol{P}$ denotes the power set operation so that $\boldsymbol{P}M$ is the class of all subsets of the class M, and $\bigcup$ the union operation: of course, $\bigcup x = \bigcup_{y \in x} y$). The terms $\breve{x}$ and $x \restriction y$ denote, respectively, the operations of forming the converse

of the relational part of the set x and of forming the relative product of the relational parts of the sets x and y. The class E is the membership relation between sets; hence, $E \restriction x = \{\langle u, v\rangle : u \in v \in x\}$. That the statement $(*)$ is equivalent to $V = L$ follows essentially from the lemma given by Hajnal ([5], p. 133) and the theorem of Shepherdson ([9], p. 186). The possibility of using the specific operations mentioned in condition (ii) of $(*)$ follows from some unpublished results of Tarski.

Let us now assume that measurable cardinals exist. Since the axiom of choice follows from $V = L$, we can identify cardinals with initial ordinals. Let $\omega_\varkappa$, then, be the least measurable cardinal. Since $2^{\aleph_0} = \aleph_1$ follows from $V = L$, we can use the arguments of [12] to conclude that $\omega_\varkappa$ must be the least 2-valued measurable cardinal and that $\omega_\varkappa$ is a strongly inaccessible number; hence, $\omega_\varkappa = \varkappa$. Let $\mu \in \{0, 1\}^{P\varkappa}$ be 2-valued, non-trivial, countably additive measure defined on all subsets of $\varkappa$. (In general if A is a class and b is a set, then A^b denotes the class of all functions with domain b and range included in A). We now employ the measure μ to define certain relations Q_μ and E_μ over the class $V^\varkappa$ as in the theory of the reduced products (ultra products) of relational systems (cf. [3] and [6]).

Definition 1.

(i) $Q_\mu = \{\langle f, g\rangle : f, g \in V^\varkappa \wedge \mu(\{\xi < \varkappa : f(\xi) = g(\xi)\}) = 1\}$;

(ii) $E_\mu = \{\langle f, g\rangle : f, g \in V^\varkappa \wedge \mu(\{\xi < \varkappa : f(\xi) \in g(\xi)\}) = 1\}$.

Lemma 1. *Q_μ is a congruence relation for E_μ over $V^\varkappa$.*

The proof is very easy and uses only the finite additivity of the measure μ. Our main interest will lie in the structure of the equivalence classes f/Q_μ under the quotient relation E_μ/Q_μ. However, the equivalence classes are not sets and the quotient relation does not really exist. The next lemma gives some facts about relation E_μ which will allow us to replace the equivalence classes by sets thus overcoming this difficulty.

Lemma 2. (i) *If $\{h \in V^\varkappa : hE_\mu f\} = \{h \in V^\varkappa : hE_\mu g\}$, then $fQ_\mu g$;*

(ii) $\{h \in V^\varkappa : hE_\mu f\} = \{h \in V^\varkappa : \exists k\,[k \in (\bigcup_{\xi<\varkappa} f(\xi) \cup \{0\})^\varkappa \wedge kE_\mu f \wedge hQ_\mu k]\}$;

(iii) $\sim \exists f\,[f \in (V^\varkappa)^\omega \wedge \forall \nu\,[\nu \in \omega \to f(\nu+1)\,E_\mu f(\nu)]]$.

Statement (i) shows that the equivalence class of f is determined by

$$\{h \in V^\varkappa : hE_\mu f\},$$

This is best proved by contradiction and requires the axiom of choice to find a function h which distinguishes f from g.

Statement (ii) implies that the number of equivalence classes included in the class $\{h \in V^\varkappa : hE_\mu f\}$ is bounded by the cardinality of the set $(\bigcup_{\xi<\varkappa} f(\xi) \cup \{0\})^\varkappa$.

Statement (iii) implies that the relation E_μ is well founded. The proof of (iii) is the first place where the countable additivity of μ is needed in the lemmas. The countable additivity at once reduces (iii) to the corresponding statement for the membership relation E, which follows easily from the axiom of foundation.

Using Lemma 2 we can now prove a statement which shows that $V^\varkappa$ can be mapped onto a class in such a way that the image of Q_μ is the identity relation and the image of E_μ is the membership relation. The method of proof is essentially that of [8] (Theorem 3, p. 147) or of [9] (Theorem 1.5, p. 171); see also [7].

LEMMA 3. *There is a (unique) function* σ *with domain* $V^\varkappa$ *such that for* $f, g \in V^\varkappa$,

(i) $\sigma(f) = \{\sigma(h) : h \in V^\varkappa \wedge hE_\mu f\}$;

(ii) $\sigma(f) = \sigma(g)$ *if and only if* $fQ_\mu g$;

(iii) $\sigma(f) \in \sigma(g)$ *if and only if* $fE_\mu g$.

DEFINITION 2. $M = \{\sigma(f) : f \in V^\varkappa\}$.

In other words, the class M is the range of the function σ; it is the class to which we shall apply the hypothesis of (*). We note first:

LEMMA 4. $M \subseteq \boldsymbol{P}M \subseteq \bigcup_{x \in M} \boldsymbol{P}x$.

The first inclusion follows at once from 3 (i) and Def. 2 To prove the second, let $y \in \boldsymbol{P}M$. Using the axiom of choice find $z \in \boldsymbol{P}(V^\varkappa)$ such that $y = \{\sigma(g) : g \in z\}$. Let $f \in V^\varkappa$ be defined so that for $\xi < \varkappa$, $f(\xi) = \{g(\xi) : g \in z\}$. Then $y \in \sigma(f)$. Before we can check the second hypothesis of (*), we need to prove a more general fact about M that can be used in many different ways. In the following $\Phi(v_0, \ldots, v_{k-1})$ will stand for any formula of set theory with free variables $v_0, \ldots, v_{k-1}$ and with *all* quantifiers restricted to V (that is, no bound class variables). Further, $\Phi^{(M)}(v_0, \ldots, v_{k-1})$ is the result of relativising all the quantifiers of $\Phi(v_0, \ldots, v_{k-1})$ to the class M.

LEMMA 5. *If* $f_0, \ldots, f_{k-1} \in V^\varkappa$, *then* $\Phi^{(M)}(\sigma(f_0), \ldots, \sigma(f_{k-1}))$ *if and only if* $\mu(\{\xi < \varkappa : \Phi(f_0(\xi), \ldots, f_{k-1}(\xi))\}) = 1$.

The proof proceeds by induction on the number of logical symbols in the formula and is exactly the same proof as that for reduced products (cf. [3], sec. 2). Now by using the proper formulas and Lemma 4 one can easily prove that M satisfies hypothesis (ii) of (*); hence, we have:

COROLLARY 5.1. $V = M$.

To obtain other corollaries, it is useful to have a short notation for the images of the constant functions in $V^\varkappa$ under the mapping σ.

DEFINITION 3. $x^* = \sigma(\{\langle \xi, x \rangle : \xi < \varkappa\})$.

COROLLARY 5.2. *If* $x_0, \ldots, x_{k-1} \in V$, *then* $\Phi^{(M)}(x_0^*, \ldots, x_{k-1}^*)$ *if and only if* $\Phi(x_0, \ldots, x_{k-1})$.

Corollary 5.2 is a direct consequence of Lemma 5 obtained by substituting the constant functions for the $f_0, \ldots, f_{k-1}$. Next, if we combine 5.1 with 5.2 using the formula $\Phi(\varkappa)$ that expresses in formal terms that $\varkappa$ is the least 2-valued measurable cardinal, we prove at once:

COROLLARY 5.3. $\varkappa = \varkappa^*$.

To show how a contradiction is reached, we introduce next a special ordinal number that does not correspond to a constant function but is the image of the identity function.

DEFINITION 4. $\delta = \sigma(\{\langle \xi, \xi \rangle : \xi < \varkappa\})$.

LEMMA 6. *If* $\lambda < \varkappa$, *then* $\lambda^* < \delta < \varkappa^*$.

Recalling that less than between ordinals is the same as membership, we see that the inequality $\delta < \varkappa^*$ follows from 3 (iii) and Definitions 3 and 4. The proof of the inequality $\lambda^* < \delta$ reduces simply to the equation $\mu(\xi < \varkappa : \lambda \leqslant \xi\}) = 1$, which follows from the fact that $\varkappa$ is the least 2-valued measurable cardinal.

Notice that from 5.2 it follows at once that the mapping from sets x to sets x^* is one-one; hence, the set $\{\lambda^* : \lambda < \varkappa\}$ must have cardinality $\varkappa$. From 6 it follows that δ must have cardinality at least that of $\varkappa$. On the other hand 5.3 and 6 together imply that $\delta < \varkappa$, which contradicts the choice of $\varkappa$ as an initial ordinal.

In case one does not wish to assume that $V = L$, the above method of proof can be used for the following definite statement: If $\varkappa$ is the least 2-valued measurable cardinal, then $\boldsymbol{PP}\varkappa \in L$.

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MATHÉMATIQUE

Problème non linéaire et discontinu d'Hilbert pour le système de fonctions

par

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Présenté par T. WAŻEWSKI le 29 avril 1961

Introduction

Soit dans le plan de la variable complexe un ensemble de p lignes fermées de Jordan $L_1, L_2, ..., L_p$ sans points communs et limitant les domaines disjoints $S_1, S_2^-, ..., S_p^-$. Soit, en outre, une ligne de Jordan L_0 embrassant toutes les lignes $L_1, L_2, ..., L_p$, mais ne les coupant pas. Toutes les lignes $L_0, L_1, ..., L_p$ ont une tangente continue. Nous posons $\sum_{\nu=0}^{p} L_\nu = L, (p \geqslant 0)$. Designons par S_0 le domaine infini situé à l'exterieur de la ligne L_0 et par S^+ — le domaine limité par la ligne L_0 et les lignes $L_1, L_2, ..., L_p$. On admet ensuite que le sens positif des lignes $L_1, L_2, ..., L_p$ est en accord avec le sens négatif de rotation dans le plan de la variable complexe et que le sens positif de la ligne L_0 est en accord avec le sens positif sur ce plan. Désignons par $c_1, c_2, ..., c_r$ une suite finie de points situés sur une au moins des lignes $L_0, L_1, ..., L_p$. Nous décomposons l'ensemble des points L en somme $L = L^{\mathrm{I}} + L^{\mathrm{II}} + L^{\mathrm{III}}$, où L^{I}, L^{II} et L^{III} sont des ensembles de points définis par les propriétés suivantes: $L_\nu \in L^{\mathrm{I}}$, si $c_\sigma \notin L_\nu$; $L_\nu \in L^{\mathrm{II}}$ s'il existe un seul point et un seulement $c_\sigma \in L_\nu$; $L_\nu \in L^{\mathrm{III}}$ si $\sum_{k=1}^{r_\nu} c_k \subset L_\nu$ $r_\nu \geqslant 2$. Observons que l'ensemble L^{III} est composé d'un nombre fini d'arcs simples $\overset{\frown}{c_\sigma c_{\sigma_1}}$. Supposons que $t, t_1 \in L_\nu \subset L^{\mathrm{I}}$, où $t, t_1 \in L_\nu \subset L_{\mathrm{II}}$ (dans ce cas nous supposons que la longueur de l'arc $\overset{\frown}{tt_1}$, $c_\sigma \notin \overset{\frown}{tt_1}$, est au plus égale à la moitié de la longueur de la ligne L_ν et $0 < |t - c_\sigma| \leqslant |t_1 - c_{\sigma_1}|$), où $t, t_1 \in L_\nu \subset L^{\mathrm{III}}$, (dans ce cas $t_1 \in \overset{\frown}{tc_\sigma}$). Nous énoncerons d'abord les définitions des classes de fonctions complexes discontinues, définies sur l'ensemble de points L, ces définitions étant données par W. Pogorzelski [1].

DÉFINITION. On appelle classe $\mathfrak{H}_a^\mu$ l'ensemble de toutes les fonctions complexes $\varphi(t)$, definies sur L qui en tout point $t \in L - \sum_{\sigma=1}^{r} c_\sigma$, vérifient les inégalités suivantes

$$(1) \qquad \prod_{\sigma=1}^{r} |t - c_\sigma|^\alpha |\varphi(t)| < \text{const}, \qquad W(t, t_1) |\varphi(t) - \varphi(t_1)| < \text{const} \, |t - t_1|^\mu,$$

où

$$W(t\,t_1) = \begin{cases} W_0 = \text{const, si } t, t_1 \in L_\nu \subset L^{\mathrm{I}}, \\ |t - c_\sigma|^{\alpha+\mu}, \text{ si } t, t_1 \in L_\nu \subset L^{\mathrm{II}},\ c_\sigma \in L_\nu, \\ |t - c_\sigma|^{\alpha+\mu}\,|t_1 - c_{\sigma_1}|^{\alpha+\mu}, \text{ si } t, t_1 \in \widehat{c_\sigma c_{\sigma_1}} \subset L^{\mathrm{III}}. \end{cases} \tag{2}$$

On admet que les paramètres réels α et μ, fixés pour la classe donnée, vérifient les inégalités $0 \leqslant \alpha < 1$, $0 < \mu < 1$, $\alpha + \mu < 1$.

Les problèmes aux limites discontinus d'Hilbert pour le système de fonctions ont été étudiés pour la première fois par J. Plemelj [2], et par N. I. Mouskhelichvili et N. P. Vécoua [3]. Dans la présente note nous allons exposer les résultats de nos recherches sur les problèmes aux limites discontinus, linéaires et non-linéaires, sous les hypothèses plus générales concernant les fonctions limites.

Problème linéaire discontinu d'Hilbert pour le système de fonctions

Enoncé du problème. Trouver le système de fonctions de variable complexe $\Phi_1(z), \Phi_2(z), \ldots, \Phi_n(z)$ dont chacune serait holomorphe dans chacun des domaines $S^+, S_0^-, \ldots, S_p^-$ séparément et dont les valeurs limites $\Phi_\nu^\pm(t)$, $\nu = 1, 2, \ldots, n$ satisferaient en tout point t de l'ensemble $L - \sum_{\sigma=1}^{r} c_\sigma$ aux relations

$$\Phi_\nu^+(t) = \sum_{\beta=1}^{n} G_{\nu\beta}(t)\,\Phi_\beta^-(t) + g_\nu(t), \tag{3}$$

où $G_{\nu\beta}(t)$ et $g_\nu(t)$ sont des fonctions données et, en outre, demandant que

$$|\Phi_\nu(t)| < \frac{\text{const}}{|z - c_\sigma|^\theta}, \qquad (\theta < 1), \tag{4}$$

au voisinage de chaque point de discontinuité. Nous supposons que:

1) Les fonctions $G_{\nu\beta}(t)$, $(\nu, \beta = 1, 2, \ldots, n)$ vérifient pour $t, t_1 \in L$ la condition de Hölder avec exposant h_G

$$|G_{\nu\beta}(t) - G_{\nu\beta}(t_1)| < k_G\,|t - t_1|^{h_G}, \qquad 0 < h_G \leqslant 1,$$

2) $\det \|G_{\nu\beta}(t)\| \neq 0$ pour $t \in L$,

3) les fonctions $g_\nu(t)$ sont de classe $\mathfrak{H}_\alpha^\mu$ (voir Introduction).

En suivant une méthode analogique à celle de N. I. Mouskhelichvili et N. P. Vécoua [3], on obtient la solution générale du problème

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_L \frac{[X^+(\tau)]^{-1}\,g(\tau)}{\tau - z}\,d\tau + X(z)\,P(z) \tag{5}$$

identique formellement avec la formule donnée dans le travail [3], où $X(z)$ désigne une matrice canonique du problème homogène (c'est à dire dans le cas $g_\nu(t) \equiv 0$), [3], $X(z) = \|\overset{\beta}{X}_\alpha(z)\|$, $\alpha, \beta = 1, 2, \ldots, n$.

$$(6) \qquad \Phi(z) = \begin{Vmatrix} \Phi_1(z) \\ \vdots \\ \Phi_n(z) \end{Vmatrix}, \quad g(t) = \begin{Vmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{Vmatrix}, \quad P(z) = \begin{Vmatrix} P_1(z) \\ \vdots \\ P_n(z) \end{Vmatrix},$$

où $P_\nu(z)$ sont certaines fonctions entières. La solution (5), d'après les propriétés des classes $\mathfrak{H}_\alpha^\mu$, vérifie l'inégalité (4). Les fonctions $\overset{\beta}{X}{}_\alpha^+(t)$ vérifient la condition de Hölder à l'exposant $\frac{1}{2} h_G$, [4].

Problème discontinu d'Hilbert généralisé

Enoncé du problème. Trouver un système de fonctions $\Phi_1(z)$, $\Phi_2(z), \ldots, \Phi_n(z)$, dont chacune serait holomorphe dans chacun des domaines $S^+, S_0^-, \ldots, S_h^-$ séparément et dont les valeurs limites $\Phi_\nu^\pm(t)$ satisferaient pour $t \in L - \sum_{\sigma=1}^{r} c_\sigma$ aux relations

$$(7) \quad \Phi_\nu^+(t) = \sum_{\beta=1}^{n} G_{\nu\beta}(t)\,\Phi_\beta^-(t) + g_\nu(t) + F_\nu[t, \Phi_1^+(t), \ldots, \Phi_n^+(t), \Phi_1^-(t), \ldots, \Phi_n^-(t)],$$

$(\nu = 1, 2, \ldots, n)$ où $G_{\nu\beta}(t)$, $g_\nu(t)$ et $F_\nu(t, u_1, \ldots, u_{2n})$ sont des fonctions données et où ces valeurs limites satisferaient à la condition (4). Nous supposons que les fonctions $G_{\nu\beta}(t)$ sont höldériennes à l'exposant h_G et det $\|G_{\nu\beta}(t)\| \neq 0$ pour $t \in L$; $g_\nu(t) \in \mathfrak{H}_\alpha^\mu$, $\alpha > 0$. Les fonctions complexes $F_\nu(t, u_1, \ldots, u_{2n})$ sont définies dans la région $t \in L$, $u_j \in \Pi$, $j = 1, 2, \ldots, 2n$, (Π désignant le plan entier de la variable complexe), et verifient l'inégalité généralisée de Hölder-Lipschitz

$$(8) \qquad |F_\nu(t, u_1, \ldots, u_{2n}) - F_\nu(t_1, u_1', \ldots, u_{2n}')| < k_F \left[\frac{|t - t_1|^{h_F}}{W(t, t_1)} + \sum_{j=1}^{2n} |u_j - u_j'| \right],$$

où $W(t, t_1)$ est définie par (2) et, de plus, ces fonctions vérifient l'inégalité suivante

$$(9) \qquad |F_\nu(t, u_1, \ldots, u_{2n})| < k_F \sum_{j=1}^{2n} |u_j| + \frac{m_F}{\prod_{\sigma=1}^{r} |t - c_\sigma|^\alpha},$$

$\nu = 1, 2, \ldots, n$, $k_F > 0$, $m_F > 0$, $0 < h_F \leqslant 1$, et $\mu < \min(\frac{1}{2} h_G, h_F)$.

Résolution du problème. En supposant que la solution $\Phi_1(z), \ldots, \Phi_n(z)$ du problème posé existe et que les fonctions limites $\Phi_\nu^\pm(t)$, $\nu = 1, 2, \ldots, 2n$, sont de classe $\mathfrak{H}_\alpha^\mu$, nous pouvons affirmer, conformément à (5), que la solution $\Phi(z) = \|\Phi_\nu(z)\|$, (voir [6]), vérifie l'égalité:

$$(10) \qquad \Phi(z) = \frac{X(z)}{2\pi i} \int_L \frac{[X^+(\tau)]^{-1} g(\tau)}{\tau - z}\, d\tau +$$

$$+ \frac{X(z)}{2\pi i} \int_L \frac{[X^+(\tau)]^{-1} F[\tau, \varphi_1(\tau), \ldots, \varphi_{2n}(\tau)]}{\tau - z}\, d\tau + X(z)\, P(z), \text{ pour } z \in S^+ + \sum_{\nu=0}^{p} S_\nu^-,$$

où $\varphi_\nu(t) = \Phi_\nu^+(t)$, $\varphi_{\nu+n}(t) = \Phi_\nu^-(t)$, $F = \|F_\nu\|$, $\nu = 1, 2, ..., n$ et les fonctions $\varphi_1(t), ..., \varphi_{2n}(t)$ forment, d'après les formules connues de Plemelj, une solution du système d'équations intégrales singulières suivant:

$$(11)\qquad \varphi_\nu(t) = f_\nu(t) + \int_L \frac{g_\nu^*(t,\tau)}{\tau - t}\,d\tau + F_\nu^*[t, \varphi_1(t), ..., \varphi_{2n}(t)] + \\ + \int_L \frac{F_\nu^{**}[t, \tau, \varphi_1(\tau), ..., \varphi_{2n}(\tau)]}{\tau - t}\,d\tau \equiv A[\varphi_1(t), ..., \varphi_{2n}(t)],$$

($\nu = 1, 2, ..., 2n$), où f_ν, g_ν^*, F_ν^* et F_ν^{**} sont des fonctions données.

Pour résoudre le système (11) nous appliquerons le théorème de Schauder [5]: Si, dans un espace de Banach, une transformation continue fait correspondre à un ensemble E de points, convexe et fermé, son sous-ensemble E' compact, il existe dans l'ensemble E au moins un point invariant de la transformation.

Considérons un espace Λ composé de tous les systèmes de $2n$ fonctions complexes $U = [\varphi_1(t), ..., \varphi_{2n}(t)]$, définies et continues dans $t \in L - \sum_{\sigma=1}^{r} c_\sigma = L_0$ qui vérifient les inégalités

$$(12)\qquad \sup_{t \in L_0} \prod_{\sigma=1}^{r} |t - c_\sigma|^{\alpha+\mu} |\varphi_\nu(t)| < \infty,$$

où α et μ (voir énoncé du problème) sont des constantes à valeur unique pour tout l'espace Λ. On définit la norme $\|U\|$ du point U, la distance entre les deux points U et V, la somme de deux points et le produit du point U par un nombre λ, par les formules suivantes

$$(13)\qquad \begin{cases} \|U\| = \max\limits_{1 \leqslant \nu \leqslant 2n} \sup\limits_{t \in L_0} \left| \prod\limits_{\sigma=1}^{r} |t - c_\sigma|^{\alpha+\mu} \varphi_\nu(t) \right|, \qquad \delta(U, V) = \|U - V\|, \\ [\varphi_1, ..., \varphi_{2n}] + [\psi_1, ..., \psi_{2n}] = [\varphi_1 + \psi_1, ..., \varphi_{2n} + \psi_{2n}], \quad \lambda U = [\lambda\varphi_1, ..., \lambda\varphi_{2n}]. \end{cases}$$

On peut démontrer que l'espace Λ est complet, donc il est un espace de Banach.

Considérons maintenant dans l'espace Λ l'ensemble E de tous les points U ayant les propriétés suivantes:

$$(14)\qquad \prod_{\sigma=1}^{r} |t - c_\sigma|^{\alpha} |\varphi_\nu(t)| \leqslant \varrho, \qquad W(t, t_1) |\varphi_\nu(t) - \varphi_\nu(t_1)| \leqslant \varkappa |t - t_1|^{\mu},$$

où ϱ et $\varkappa$ sont des nombres positifs fixés arbitrairement. L'ensemble E est évidemment fermé et convexe.

En tenant compte de la forme du système d'équations proposé (11), transformons l'ensemble E à l'aide des relations

$$(15)\qquad \psi_\nu(t) = A[\varphi_1(t), ..., \varphi_{2n}(t)], \qquad (\nu = 1, 2, ..., n).$$

Nous allons énoncer les lemmes suivants:

LEMME 1. *L'ensemble E' transformé à l'aide de la relation* (15) *est un sous-ensemble de l'ensemble E, si k_F est suffisamment petit.*

LEMME 2. *La transformation de l'ensemble E à l'aide de la relation* (15) *est continue dans l'espace* Λ.

LEMME 3. *L'ensemble E' est compact.*

Par conséquent, il existe au moins une solution du système (11) composée des fonctions de classe $\mathfrak{H}_a^{\mu}$. Nous pouvons donc énoncer le théorème suivant.

THÉORÈME. *Si les fonctions données* $G_{\nu\beta}$, F_ν, g_ν *vérifient les hypothèses données dans l'énoncé du problème et* k_F *est suffisamment petit, alors il existe au moins une solution* $\Phi_1(z), \ldots, \Phi_n(z)$ *du problème posé; toutes les solutions sont déterminées par les formules* (10), *où* $P_\nu(z)$ *sont des fonctions entières arbitraires et* $\varphi_1(t), \ldots, \varphi_{2n}(t)$ *constituent les solutions du système d'équations intégrales* (11).

La démonstration du Théorème et des Lemmes sera publiée dans un travail qui paraîtra sous le même titre dans le Bulletin de la Haute École Militaire des Sciences Techniques.

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MATHEMATICS

On Solutions of Non-Linear Parabolic Equations Defined in Unbounded Domains

by

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Presented by T. WAŻEWSKI on May 12, 1961

Let us consider the system of non-linear equations of the form

$$(1) \qquad \frac{\partial u_s}{\partial t} = F_s\left(x, t, u_i, \frac{\partial u_s}{\partial x_j}, \frac{\partial^2 u_s}{\partial x_j \partial x_k}\right)\begin{pmatrix} s, i = 1, \ldots, n \\ j, k = 1, \ldots, m \end{pmatrix}, \quad x = (x_1, \ldots, x_m),$$

where the derivatives of functions $u_1, \ldots, u_{s-1}, u_{s+1}, \ldots, u_n$ do not appear in the s-th equation. In this note certain theorems (which together with their proofs will appear in the Annales Polonici Mathematici) concerning the solutions of Fourier's problems in unbounded domains for the system (1) are formulated. These problems have been investigated by J. Szarski in [5], [6], but in domains whose intersections with planes $t = \text{const}$ are bounded. In case of one parabolic linear equation theorems analogous to Theorems 1.3 and 4 contained here have been formulated and proved by M. Krzyżański [1]—[3]. Theorems similar to 2 and 5 have been treated by W. Mlak [4] for strong inequalities in bounded domains.

1. Denote by D an open unbounded domain of Euclidean space E^{m+1} of the variables $x_1, \ldots, x_m, t$ contained between planes $t = 0$ and $t = T > 0$ whose boundary consists of unbounded domains S^0, S^T lying on planes $t = 0$, $t = T$, respectively, and of a certain surface σ nontangent to any plane $t = \text{const}$.

We apply the following definition given by J. Szarski: let

$$(2) \qquad w_1(x, t), \ldots, w_n(x, t)$$

be a sequence of functions of the class C^1 in D. An equation with the index s of (1) is called parabolic with respect to the sequence (2) if for every two systems of numbers $z_{jk}, \dot{z}_{jk}$ $(j, k = 1, \ldots, m)$, $z_{jk} = z_{kj}$, $\dot{z}_{jk} = \dot{z}_{kj}$, such that $\sum_{j,k=1}^{m} (z_{jk} - \dot{z}_{jk}) \lambda_j \lambda_k \leqslant 0$ for every vector $(\lambda_1, \ldots, \lambda_m)$, the inequality

$$F_s\left(x, t, w_i(x, t), \frac{\partial w_s(x, t)}{\partial x_j}, z_{jk}\right) - F_s\left(x, t, w_i(x, t), \frac{\partial w_s(x, t)}{\partial x_j}, \dot{z}_{jk}\right) \leqslant 0$$

holds. If every equation of the system (1) is parabolic with respect to the solution $u_i(x, t)$ $(i = 1, \ldots, n)$ of this system, such a solution is called a parabolic solution.

A solution $u_i(x, t)$ of (1) is said to be regular in D if the functions $u_i(x, t)$ are continuous in the closure $\bar{D}$ of domain D and if they possess the derivative $\partial u_s/\partial t$ as well as continuous derivatives $\dfrac{\partial^2 u_s}{\partial x_j \partial x_k}$ $(s = 1, ..., n;\ j, k = 1, ..., m)$ in the interior of D.

Let $\varphi_i(x, t)$ $(i = 1, ..., n)$ be given functions, defined and continuous on the set $\sum \overset{\text{df}}{=} S^0 + \sigma$. The problem which we will call (FI) is formulated as follows: find the parabolic solution $u_i(x, t)$ $(i = 1, ..., n)$ of system (1), regular in D and fulfilling the condition

$$u_i(x, t) = \varphi_i(x, t) \ (i = 1, ..., n) \text{ for } (x, t) \in \sum.$$

We will say that the function $F_s(x, t, y_i, z_j, z_{jk})$ satisfies the $(\mathcal{L})$-condition if there exist positive constants $L_0, ..., L_4$ such that for arbitrary y_i, z_j, z_{jk}; $\bar{y}_i, \bar{z}_j, \bar{z}_{jk}$ $(i = 1, ..., n;\ j, k = 1, ..., m)$, $y_s \geqslant \bar{y}_s$, we have the inequality

$$F_s(x, t, y_i, z_j, z_{jk}) - F_s(x, t, \bar{y}_i, \bar{z}_j, \bar{z}_{jk})$$

$$\leqslant L_0 \sum_{j,k=1}^{m} |z_{jk} - \bar{z}_{jk}| + (L_1 |x| + L_2) \sum_{j=1}^{m} |z_j - \bar{z}_j| + (L_3 |x|^2 + L_4) \sum_{i=1}^{n} |y_i - \bar{y}_i|,$$

where $|x| = \left(\sum_{i=1}^{m} x_i^2 \right)^{1/2}$.

By E_2 we denote the class of functions $\psi(x, t)$, defined in an unbounded domain, for which there exist positive constants M, K such that

$$|\psi(x, t)| \leqslant M \exp(K |x|^2)$$

in this domain.

THEOREM 1. *If each function $F_s(x, t, y_i, z_j, z_{jk})$ $(s = 1, ..., n)$ satisfies the condition $(\mathcal{L})$, then the (FI)-problem possesses no more than one solution of class E_2 in the domain D.*

Let us take now a system of functions $\chi_s(y_1, ..., y_n, \tau)$ $(s = 1, ..., n)$, where τ denotes a sequence of variables different from $y_1, ..., y_n$. We will say that the function χ_s fulfils the (W)-condition with respect to the variables $y_1, ..., y_n$ if for $y_i \leqslant \bar{y}_i$, $i \neq s$, $y_s = \bar{y}_s$, we have the inequality

$$\chi_s(y_1, ..., y_n, \tau) \leqslant \chi_s(\bar{y}_1, ..., \bar{y}_n, \tau).$$

THEOREM 2. *If*

1° *$u_i(x, t)$, $v_i(x, t)$ $(i = 1, ..., n)$ are regular solutions of class E_2 in D of the system of equations*

$$\frac{\partial u_s}{\partial t} = F_s^{(1)}\left(x, t, u_i, \frac{\partial u_s}{\partial x_j}, \frac{\partial^2 u_s}{\partial x_j \partial x_k}\right), \tag{3}$$

$(s = 1, ..., n)$

$$\frac{\partial v_s}{\partial t} = F_s^{(2)}\left(x, t, v_i, \frac{\partial v_s}{\partial x_j}, \frac{\partial^2 v_s}{\partial x_j \partial x_k}\right), \tag{4}$$

respectively;

2° *for each* s $(s = 1, ..., n)$ *the equation with index* s *of system* (3) *is parabolic with respect to the sequence* $u_i(x, t)$ *or the equation with the same index of system* (4) *is parabolic with respect to the sequence* $v_i(x, t)$;

3° *for each* s *the function* $F_s^{(1)}$ *or* $F_s^{(2)}$ *satisfies the* (W)*-condition with respect to the variables* $y_1, ..., y_n$ *and the* $(\mathcal{L})$*-condition*;

4° $F_s^{(1)}(x, t, y_i, z_j, z_{jk}) \leqslant F_s^{(2)}(x, t, y_i, z_j, z_{jk})$ $(s = 1, ..., n)$ *in the domain of existence of these functions*;

5° $u_i(x, t) \leqslant v_i(x, t)$ $(i = 1, ..., n)$ *for* $(x, t) \in \Sigma$,

then the inequalities

$$u_i(x, t) \leqslant v_i(x, t) \quad (i = 1, ..., n)$$

are satisfied everywhere in $\bar{D}$.

Let D_R denote the part of the domain D contained inside the cylindric surface Γ_R^* with the equation $\sum_{i=1}^{m} x_i^2 = R^2$. Further, put $C_R = \bar{D} \cdot \Gamma_R$, $\Sigma_R^* = \bar{D}_R \cdot \Sigma$. Let $\Phi_i(x, t)$ $(i = 1, ..., n)$ be continuous functions of class E_2 in $\bar{D}$.

THEOREM 3. *If*

1° *for arbitrary* $\Phi_i(x, t)$ *and for every* $R > R_0$ *there exists a parabolic solution of* (1), *regular in* D_R *and satisfying* $u_i(x, t) = \Phi_i(x, t)$ $(i = 1, ..., n)$ *on the set* $\Sigma_R + C_R$;

2° *the functions* $F_s(x, t, y_i, z_j, z_{jk})$ $(s = 1, ..., n)$ *satisfy the* $(\mathcal{L})$*-condition*;

3° *the functions* $F_s(x, t, 0, 0, 0)$ $(s = 1, ..., n)$ *belong to* E_2;

4° $\varphi_i(x, t)$ $(i = 1, ..., n)$ *are given functions defined, continuous and belonging to* E_2 *in* Σ;

5° *the number* T *is sufficiently small;*

then there exists a solution of the (FI)*-problem and this solution belongs to* E_2 *in* D.

If the domain D is an unbounded zone

$$D\{0 < t < T, -\infty < x_i < +\infty \ (i = 1, ..., m)\},$$

then the set σ should be considered as empty and the Theorems 1—3 remain true for Cauchy's problem.

2. Let Δ be a bounded and closed domain situated in m-dimensional Euclidean space E^m of the variables $x_1, ..., x_m$ and let S^0 be its complementary domain. We assume that the boundary $F(\Delta)$ of Δ may be represented by the equation $G(x) = 0$, where $G(x)$ is a function with continuous and bounded derivatives of the second order in the domain S^0, while it is of class C^1 in the closure $\bar{S}^0$ and satisfies the condition

$$|\operatorname{grad} G(x)| \geqslant \Gamma > 0.$$

Now, we define $D = S^0 \times (0, T)$, $\sigma = F(\Delta) \times \langle 0, T \rangle$, $T > 0$.

Let $\varphi_i(x)$ $(i = 1, ..., n)$ be arbitrarily given functions defined and continuous for $x \in S^0$, while $G_s(x, t, y_1, ..., y_n)$ $(s = 1, ..., n)$ are defined for $(x, t) \in \sigma$, $-\infty < y_i < +\infty$.

For every $(x, t) \in \sigma$ and every s $(s = 1, ..., n)$ let l_s be a straight half-line entering into the interior of D (at the point (x, t)) and parallel to the plane $t = 0$. We assume that there exists a positive constant γ_0 such that $\cos(l_i, n_0) \geqslant \gamma_0 > 0$ $(i = 1, ..., n)$ for $(x, t) \in \sigma$, where n_0 denotes the normal to σ directed to the interior of D.

The following problem is called the (F)-problem: to find a parabolic solution $u_i(x, t)$ $(i = 1, ..., n)$ of (1), regular in D, possessing the derivatives $\frac{du_s}{dl_s}$ $(s = 1, ..., n)$ at points of σ, and fulfilling the initial condition

$$u_i(x, 0) = \varphi_i(x) \quad (i = 1, ..., n) \text{ for } x \in S^0$$

as well as the boundary condition

$$\frac{du_s}{dl_s} + G_s(x, t, u_1, ..., u_n) = 0 \quad (s = 1, ..., n) \text{ for } (x, t) \in \sigma.$$

We will say that the function G_s satisfies the condition $(\mathcal{L})$ if there exists a positive constant $\bar{L}$ such that for arbitrary y_i, $\bar{y}_i$ $(i = 1, ..., n)$, $y_s \geqslant \bar{y}_s$, the inequality

$$G_s(x, t, y_1, ..., y_n) - G_s(x, t, \bar{y}_1, ..., \bar{y}_n) \leqslant L \sum_{i=1}^{m} |y_i - \bar{y}_i|$$

holds.

THEOREM 4. *If the functions F_s and G_s fulfil the conditions $(\mathcal{L})$ and $(\bar{\mathcal{L}})$, respectively, then the problem (F) possesses no more than one solution belonging to class E_2 in the domain D.*

THEOREM 5. *If the assumptions* 1°—4° *of Theorem* 2 *are satisfied, and if*

5° $u_i(x, 0) \leqslant v_i(x, 0)$ $(i = 1, ..., n)$ *for* $x \in S^0$;

6° *the functions* $u_i(x, t)$ *and* $v_i(x, t)$ *fulfil the boundary conditions*

$$\frac{du_s}{dl_s} + G_s^{(1)}(x, t, u_1, ..., u_n) = 0$$

$$\frac{dv_s}{dl_s} + G_s^{(2)}(x, t, v_1, ..., v_n) = 0,$$

$(s = 1, ..., n)$ *for* $(x, t) \in \sigma$

respectively;

7° $G_s^{(1)}(x, t, y_1, ..., y_n) \leqslant G_s^{(2)}(x, t, y_1, ..., y_n)$ $(s = 1, ..., n)$ *for* $(x, t) \in \sigma$, $-\infty < y_i < +\infty$;

8° *for each* s $(s = 1, ..., n)$ *the function* $G_s^{(1)}$ *or the function* $G_s^{(2)}$ *fulfils the condition* $(\bar{\mathcal{L}})$ *and the condition* (W) *with respect to the variables* $y_1, ..., y_n$.

then the inequalities

$$u_i(x, t) \leqslant v_i(x, t) \quad (i = 1, ..., n)$$

are satisfied everywhere in D.

Let us choose R_1 such that for $R > R_1$ the surface σ should lie inside the surface Γ_R with the equation $\sum_{i=1}^{m} x_i^2 = R^2$. Put $C_R = D \cdot \Gamma_R$. We denote by D_R and S_R^0 the parts of domains D and S^0 situated inside Γ_R, respectively.

Let $\Phi_i(x, t)$ $(i = 1, ..., n)$ be arbitrary continuous functions of class E_2 in $\bar{D}$, and let $G_s(x, t, y_1, ..., y_n)$ be given functions defined for $(x, t) \in \sigma$, $-\infty < y_i < +\infty$ and fulfilling the condition $(\bar{\mathcal{L}})$.

THEOREM 6. *If*

1° *for arbitrary* $\Phi_i(x, t)$ *and for every* $R > R_1$ *there exists a parabolic solution* $u_i(x, t)$ *of* (1), *regular in* D_R *and fulfilling the conditions*

$$u_i(x, t) = \Phi_i(x, t) \ (i = 1, ..., n) \quad \text{on the set } S_R^0 + C_R,$$

$$\frac{du_s}{dl_s} + G_s(x, t, u_1, ..., u_n) = 0 \qquad (s = 1, ..., n) \text{ for } (x, t) \in \sigma;$$

2° *the functions* F_s *and* G_s $(s = 1, ..., n)$ *satisfy the conditions* $(\mathcal{L})$ *and* $(\bar{\mathcal{L}})$, *respectively*;

3° *the functions* $F_s(x, t, 0, 0, 0)$ *belong to* E_2 *in* D, *while* $G_s(x, t, 0, ..., 0)$ *are bounded for* $(x, t) \in \sigma$;

4° $\varphi_i(x)$ *are given continuous functions of class* E_2 *in* S^0;

5° *the number* T *is sufficiently small,*

then there exists a solution of the problem (F) *and this solution belongs to* E_2 *in* D.

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Representations of the Space Group of Rutile (TiO_2)

by

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Symmetry of rutile

The unit cell for rutile [1] is illustrated in Figs. 1 and 2. The rutile structure is based on the simple tetragonal lattice (Γ_q). The three basic primitive translations are:

$$\boldsymbol{t}_1 = (a, 0, 0),\ \boldsymbol{t}_2 = (0, a, 0),\ \boldsymbol{t}_3 = (0, 0, c).$$

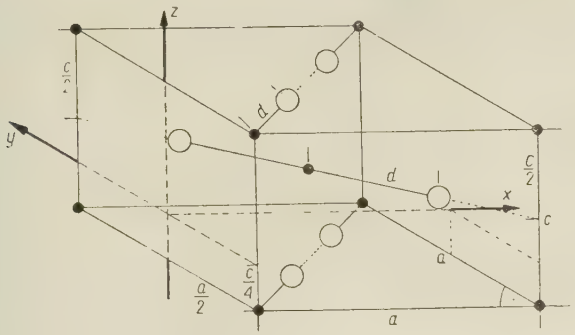

Fig. 1. ● — atoms of Ti, ○ — atoms of O. The lattice constants are: $a = 4.58$ Å, $c = 2.95$ Å, the ratio $c/a = 0.644$. The Ti—O distance $d = 2.01$ Å.

The space group of rutile is D_{4h}^{14}, that is $P\frac{4_2}{n}\frac{2_1}{n}\frac{2}{m}$. Its generators in our co-ordinate system are:

$$[\varepsilon\,|\,\Gamma_q],\quad \left[C_4\,\middle|\,\frac{\boldsymbol{t}_3}{2}\right],\quad \left[C_2'\,\middle|\,\frac{\boldsymbol{t}_1+\boldsymbol{t}_2}{2}\right],\quad \left[I\,\middle|\,\frac{\boldsymbol{t}_3}{2}\right].$$

We have chosen the form of the generators following Seitz [2], except that our distinguished axis is the z-axis and not the x-axis. The further elements of the group are:

$$[C_2\,|\,0],\quad \left[C_4^3\,\middle|\,\frac{\boldsymbol{t}_3}{2}\right],\quad \left[C_2''\,\middle|\,\frac{\boldsymbol{t}_1+\boldsymbol{t}_2}{2}\right],\quad \left[C_2^d\,\middle|\,\frac{\boldsymbol{t}_1+\boldsymbol{t}_2+\boldsymbol{t}_3}{2}\right],\quad \left[C_2^{d'}\,\middle|\,\frac{\boldsymbol{t}_1+\boldsymbol{t}_2+\boldsymbol{t}_3}{2}\right],$$

$$[S_4^3|0],\quad \left[\sigma_h\left|\frac{\boldsymbol{t}_3}{2}\right.\right],\quad [S_4|0],\quad \left[\sigma_v\left|\frac{\boldsymbol{t}_1+\boldsymbol{t}_2+\boldsymbol{t}_3}{2}\right.\right],\quad \left[\sigma_v'\left|\frac{\boldsymbol{t}_1+\boldsymbol{t}_2+\boldsymbol{t}_3}{2}\right.\right],$$

$$\left[\sigma_d\left|\frac{\boldsymbol{t}_1+\boldsymbol{t}_2}{2}\right.\right],\quad \left[\sigma_d'\left|\frac{\boldsymbol{t}_1+\boldsymbol{t}_2}{2}\right.\right]$$

and the products of the primitive translations by all listed elements.

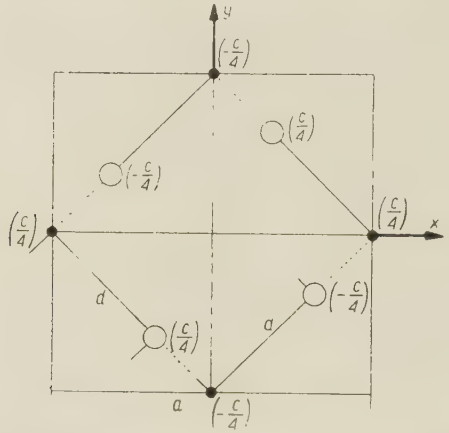

Fig. 2. The projection of the rutile structure into the xy plane. The heights of the lattice points above the $z = 0$ plane are given at each atom

We use for brevity:

$$\varepsilon = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad C_4 = \begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix},\quad C_2 = \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},\quad C_4^3 = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix},$$

$$C_2' = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\quad C_2'' = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},\quad C_2^d = \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix},\quad C_2^{d'} = \begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix},$$

$$I = \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\quad S_4^3 = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&-1\end{pmatrix},\quad \sigma_h = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix},\quad S_4 = \begin{pmatrix}0&-1&0\\1&0&0\\0&0&-1\end{pmatrix},$$

$$\sigma_v = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \sigma_v' = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix},\quad \sigma_d = \begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix},\quad \sigma_d' = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

To take into account the spin-orbit coupling we must consider the so-called double groups. The two-dimensional matrices ([3], p. 181) corresponding to the elements of the double group D_{4h} are given by formula:

$$D_{1/2}(\psi, \varphi, \theta) = \begin{pmatrix}\alpha&\beta\\-\beta^*&\alpha^*\end{pmatrix},$$

where $$\alpha = e^{\frac{i}{2}(\psi+\varphi)} \cos\frac{\theta}{2}, \qquad \beta = -ie^{\frac{i}{2}(\psi-\varphi)} \sin\frac{\theta}{2},$$

ψ, φ, θ — Euler's angles as defined by Lubarski ([3], p. 16).

For unbarred operators we have assumed the restrictions:

$$0 \leqslant \psi + \varphi < 2\pi, \text{ if } \theta = 0; \; 0 \leqslant \psi - \varphi < 2\pi, \text{ if } \theta = \pi.$$

Of course, $D_{1/2}(\bar{\alpha}) = -D_{1/2}(\alpha)$.

Representations

The basic vectors for our reciprocal lattice are:

$$\boldsymbol{K}_1 = 2\pi\left(\frac{1}{a}, 0, 0\right), \quad \boldsymbol{K}_2 = 2\pi\left(0, \frac{1}{a}, 0\right), \quad \boldsymbol{K}_3 = 2\pi\left(0, 0, \frac{1}{c}\right).$$

The symmetrical unit cell for the reciprocal lattice (i.e. the first Brillouin Zone) is illustrated in Fig. 3:

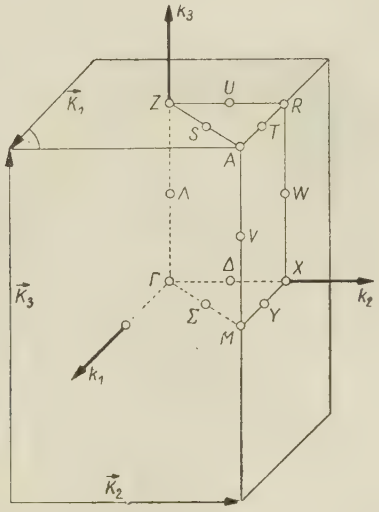

Fig. 3. The first Brillouin Zone for the simple tetragonal lattice

To give the representations of a space group it is sufficient [4] to know the representations of corresponding groups $G(\boldsymbol{k})$ of wave vector $\boldsymbol{k}$, where $\boldsymbol{k}$ wanders over the interior and surface of the first Brillouin Zone.

Let us denote by $G_0(\boldsymbol{k})$ a (point) group, consisting of the rotational parts of those elements $[\alpha|\boldsymbol{v}(\alpha)]$, which form the group $G(\boldsymbol{k})$ of given wave vector $\boldsymbol{k}$.

Representations for the corresponding double groups $G_0(\boldsymbol{k})$ have been given in the Koster's article [4]. To obtain the matrices corresponding to an element $[\alpha|\boldsymbol{v}(\alpha)]$ of $G(\boldsymbol{k})$ for $\boldsymbol{k}$ within Brillouin Zone one has to multiply the matrices corresponding to element α of $G_0(\boldsymbol{k})$ by factors $\exp[i\boldsymbol{k}\boldsymbol{v}(\alpha)]$, where $\boldsymbol{k}$ is the vector under consideration.

For the points on the surface of Brillouin Zone we follow the method of Lubarski ([3], p. 88):

For each ordered pair of elements of a double group $G_0(\boldsymbol{k})$ we introduce the weight function defined by:

$$w(\alpha_1, \alpha_2) = \exp^{[i(\alpha_1^{-1}\boldsymbol{k}-\boldsymbol{k})\boldsymbol{v}_2]},$$

where $[\alpha|\boldsymbol{v}_1]$, $[\alpha_2|\boldsymbol{v}_2]$ are elements of the group $G(\boldsymbol{k})$. We give in the tables the matrices $\hat{D}(\alpha)$ of the "weighted" representations of groups $G_0(\boldsymbol{k})$, i.e., which obey the multiplication rule

$$\hat{D}(\alpha_1)\hat{D}(\alpha_2) = w(\alpha_1, \alpha_2)\hat{D}(\alpha_1, \alpha_2).$$

For the representations above the line $\hat{D}(\overline{\alpha}) = \hat{D}(\alpha)$; below the line we listed the so-called additional representations, for which $\hat{D}(\overline{\alpha}) = -\hat{D}(\alpha)$.

To obtain the usual (i.e. without weight) representations of $G(\boldsymbol{k})$ it is necessary to multiply the matrices given in our tables by factors $\exp[i\boldsymbol{k}\boldsymbol{v}(\alpha)]$, where $\boldsymbol{k}$ is the vector under consideration.

The representations given in the tables are, of course, arbitrarily chosen from the sets of equivalent (unitary) representations.

In the tables $\omega = e^{i\frac{\pi}{4}}$.

TABLE I

The plane US, $\boldsymbol{k} = \left(k_x, k_y, \frac{\pi}{c}\right)$

	ε	σ_h
1	1	i
2	1	$-i$
3	1	1
4	1	-1

TABLE II

The plane VW, $\boldsymbol{k} = \left(k_x, \frac{\pi}{a}, k_z\right)$

	ε	σ_v'
1	1	i
2	1	$-i$
3	1	1
4	1	-1

TABLE III

The axis Y, $\boldsymbol{k} = \left(k_x, \frac{\pi}{a}, 0\right)$

	ε	C_2'	σ_h	σ_v
1	1	i	1	i
2	1	$-i$	1	$-i$
3	1	i	-1	$-i$
4	1	$-i$	-1	i
5	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$

TABLE IV

The axis S, $\boldsymbol{k} = \left(k_x, k_x, \frac{\pi}{c}\right)$

	ε	C_2^d	σ_h	σ_d
1	1	i	i	1
2	1	$-i$	$-i$	1
3	1	i	$-i$	-1
4	1	$-i$	i	-1
5	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

TABLE V

The axis T, $\boldsymbol{k} = \left(k_x, \frac{\pi}{a}, \frac{\pi}{c}\right)$

	ε	C_2'	σ_h	σ_v'
1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$
2	1	1	1	1
3	1	−1	1	−1
4	1	1	−1	−1
5	1	−1	−1	1

TABLE VI

The axis U, $\boldsymbol{k} = \left(0, k_y, \frac{\pi}{c}\right)$

	ε	C_2''	σ_h	σ_v
1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
2	1	i	1	$-i$
3	1	$-i$	1	i
4	1	i	−1	i
5	1	$-i$	−1	$-i$

TABLE VII

The axis W, $\boldsymbol{k} = \left(0, \frac{\pi}{a}, k_z\right)$

	ε	C_2	σ_v	σ_v'
1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$
2	1	i	i	1
3	1	i	$-i$	−1
4	1	$-i$	i	−1
5	1	$-i$	$-i$	1

TABLE VIII

The axis V, $\boldsymbol{k} = \left(\frac{\pi}{a}, \frac{\pi}{a}, k_z\right)$

	ε	C_4	C_2	C_4^3	σ_v	σ_v'	σ_d	σ_d'
1	1	i	−1	$-i$	$-i$	$-i$	1	−1
2	1	i	−1	$-i$	$-i$	i	−1	1
3	1	$-i$	−1	i	i	$-i$	−1	1
4	1	$-i$	−1	i	$-i$	i	1	−1
5	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
6	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}\omega&0\\0&-\omega^*\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-\omega^*&0\\0&\omega\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega\\\omega^*&0\end{pmatrix}$	$\begin{pmatrix}0&\omega^*\\-\omega&0\end{pmatrix}$
7	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-\omega&0\\0&\omega^*\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}\omega^*&0\\0&-\omega\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&\omega\\-\omega^*&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega^*\\\omega&0\end{pmatrix}$

TABLE IX

The point $X, k = \left(0, \frac{\pi}{a}, 0\right)$

	ε	C_2	C_2'	C_2''	I	σ_h	σ_v	σ_v'
1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

TABLE X

The point $R, k = \left(0, \frac{\pi}{a}, \frac{\pi}{c}\right)$

	ε	C_2	C_2'	C_2''	I	σ_h	σ_v	σ_v'
1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$
3	1	i	1	$-i$	i	-1	$-i$	-1
4	1	i	1	$-i$	$-i$	1	i	1
5	1	i	-1	i	i	-1	i	1
6	1	i	-1	i	$-i$	1	$-i$	-1
7	1	$-i$	1	i	i	1	$-i$	1
8	1	$-i$	1	i	$-i$	-1	i	-1
9	1	$-i$	-1	$-i$	i	1	i	-1
10	1	$-i$	-1	$-i$	$-i$	-1	$-i$	1

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TABLE XI

The point M, $\boldsymbol{k} = \left(\frac{\pi}{a}, \frac{\pi}{a}, 0\right)$

	ε	C_4	C_2	C_4^3	C_2'	C_2''	C_2^d	$C_2^{d'}$	I	S_4^3	σ_v	S_4	σ_d	C_4	σ_v	C_4^3
1	1	i	-1	$-i$	i	$-i$	1	-1	1	i	-1	$-i$	i	-1	1	-1
2	1	i	-1	$-i$	$-i$	i	-1	1	1	i	-1	$-i$	$-i$	1	-1	1
3	1	i	-1	$-i$	i	$-i$	1	-1	-1	$-i$	1	i	$-i$	1	-1	1
4	1	i	-1	$-i$	$-i$	i	-1	1	-1	$-i$	1	i	i	-1	1	-1
5	1	$-i$	-1	i	i	$-i$	-1	1	1	$-i$	-1	i	i	-1	-1	1
6	1	$-i$	-1	i	$-i$	i	1	-1	1	$-i$	-1	i	$-i$	1	1	-1
7	1	$-i$	-1	i	i	$-i$	-1	1	-1	i	1	$-i$	$-i$	1	1	-1
8	1	$-i$	-1	i	$-i$	i	1	-1	-1	i	1	$-i$	$-i$	1	-1	1
9	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$
10	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
11	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}\omega&0\\0&-\omega^*\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-\omega^*&0\\0&\omega\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega\\\omega^*&0\end{pmatrix}$	$\begin{pmatrix}0&\omega^*\\-\omega&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}\omega&0\\0&-\omega^*\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-\omega^*&0\\0&\omega\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega\\\omega^*&0\end{pmatrix}$	$\begin{pmatrix}0&\omega^*\\-\omega&0\end{pmatrix}$
12	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}\omega^*&0\\0&-\omega\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}-\omega&0\\0&\omega^*\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega^*\\\omega&0\end{pmatrix}$	$\begin{pmatrix}0&\omega\\-\omega^*&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}\omega^*&0\\0&-\omega\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}-\omega&0\\0&\omega^*\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega^*\\\omega&0\end{pmatrix}$	$\begin{pmatrix}0&\omega\\-\omega^*&0\end{pmatrix}$
13	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}\omega&0\\0&-\omega^*\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-\omega^*&0\\0&\omega\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega\\\omega^*&0\end{pmatrix}$	$\begin{pmatrix}0&\omega^*\\-\omega&0\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-\omega&0\\0&\omega^*\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}\omega^*&0\\0&-\omega\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&\omega\\-\omega^*&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega^*\\\omega&0\end{pmatrix}$
14	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}\omega^*&0\\0&-\omega\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}-\omega&0\\0&\omega^*\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega^*\\\omega&0\end{pmatrix}$	$\begin{pmatrix}0&\omega\\-\omega^*&0\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-\omega^*&0\\0&\omega\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}\omega&0\\0&-\omega^*\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&\omega^*\\-\omega&0\end{pmatrix}$	$\begin{pmatrix}0&-\omega\\\omega^*&0\end{pmatrix}$

TABLE XII

The point Z, $\boldsymbol{k} = \left(0,\ 0,\ \frac{\pi}{c}\right)$

	ε	C_4	C_2	C_4^3	C_2'	C_2''	C_2^d	$C_2^{d'}$	I	S_4^3	σ_h	S_4	σ_v	σ_v'	σ_d	σ_d'
1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
2	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$
3	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$
4	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$
5	$\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix}$	$\begin{matrix}\omega&0&0&0\\0&-\omega&0&0\\0&0&\omega^*&0\\0&0&0&-\omega^*\end{matrix}$	$\begin{matrix}i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&-i\end{matrix}$	$\begin{matrix}-\omega^*&0&0&0\\0&\omega^*&0&0\\0&0&-\omega&0\\0&0&0&\omega\end{matrix}$	$\begin{matrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{matrix}$	$\begin{matrix}0&0&0&i\\0&0&i&0\\0&i&0&0\\i&0&0&0\end{matrix}$	$\begin{matrix}0&0&0&\omega\\0&0&-\omega&0\\0&-\omega^*&0&0\\\omega^*&0&0&0\end{matrix}$	$\begin{matrix}0&0&0&-\omega^*\\0&0&\omega^*&0\\0&\omega&0&0\\-\omega&0&0&0\end{matrix}$	$\begin{matrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{matrix}$	$\begin{matrix}0&\omega&0&0\\\omega&0&0&0\\0&0&0&\omega^*\\0&0&\omega^*&0\end{matrix}$	$\begin{matrix}0&i&0&0\\-i&0&0&0\\0&0&0&-i\\0&0&i&0\end{matrix}$	$\begin{matrix}0&-\omega^*&0&0\\-\omega^*&0&0&0\\0&0&0&-\omega\\0&0&-\omega^*&0\end{matrix}$	$\begin{matrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{matrix}$	$\begin{matrix}0&0&i&0\\0&0&0&-i\\i&0&0&0\\0&-i&0&0\end{matrix}$	$\begin{matrix}0&0&\omega&0\\0&0&0&\omega\\\omega^*&0&0&0\\0&\omega^*&0&0\end{matrix}$	$\begin{matrix}0&0&-\omega^*&0\\0&0&0&-\omega^*\\\omega&0&0&0\\0&\omega&0&0\end{matrix}$

TABLE XIII

The point A, $\boldsymbol{k} = \left(\frac{\pi}{a},\ \frac{\pi}{a},\ \frac{\pi}{c}\right)$

	ε	C_4	C_2	C_4^3	C_2'	C_2''	C_2^d	$C_2^{d'}$	I	S_4^3	σ_h	S_4	σ_v	σ_v'	σ_d	σ_d'
1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$
2	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\i&0\end{pmatrix}$
3	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$
4	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&i\\i&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$	$\begin{pmatrix}-i&0\\0&i\end{pmatrix}$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
5	$\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix}$	$\begin{matrix}\omega&0&0&0\\0&-\omega&0&0\\0&0&-\omega^*&0\\0&0&0&\omega^*\end{matrix}$	$\begin{matrix}i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&-i\end{matrix}$	$\begin{matrix}-\omega^*&0&0&0\\0&\omega^*&0&0\\0&0&\omega&0\\0&0&0&-\omega\end{matrix}$	$\begin{matrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{matrix}$	$\begin{matrix}0&0&0&i\\0&0&i&0\\0&i&0&0\\i&0&0&0\end{matrix}$	$\begin{matrix}0&0&0&-\omega\\0&0&\omega&0\\0&\omega^*&0&0\\\omega^*&0&0&0\end{matrix}$	$\begin{matrix}0&0&0&\omega^*\\0&0&-\omega^*&0\\0&-\omega&0&0\\\omega&0&0&0\end{matrix}$	$\begin{matrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{matrix}$	$\begin{matrix}0&\omega&0&0\\\omega&0&0&0\\0&0&0&-\omega^*\\0&0&-\omega^*&0\end{matrix}$	$\begin{matrix}0&i&0&0\\-i&0&0&0\\0&0&0&-i\\0&0&i&0\end{matrix}$	$\begin{matrix}0&-\omega^*&0&0\\-\omega^*&0&0&0\\0&0&0&\omega\\0&0&\omega&0\end{matrix}$	$\begin{matrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{matrix}$	$\begin{matrix}0&0&i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{matrix}$	$\begin{matrix}0&0&-\omega&0\\0&0&0&-\omega\\-\omega^*&0&0&0\\0&\omega^*&0&0\end{matrix}$	$\begin{matrix}0&0&\omega^*&0\\0&0&0&\omega^*\\-\omega&0&0&0\\0&-\omega&0&0\end{matrix}$

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Geometrical and Physical Interpretation of the Weyl Conformal Curvature Tensor

by

F. A. E. PIRANI and A. SCHILD

Presented by L. INFELD on May 13, 1961

1. Introduction: Conformal-invariant methods

This paper outlines, without proofs, a geometrical and physical interpretation of the Weyl tensor in the space-time of general relativity. Proofs and further details will be published elsewhere. The interpretation embodies a method whereby the physical components of the Weyl tensor could, in principle, be measured by observations of light-rays alone, without the use of clocks or rigid rods. The possibility of such measurements may be inferred from the fact that null geodesics, which represent light rays, are invariant under conformal transformations of Riemannian space-time, while proper time along time-like lines is not invariant (nor are time-like geodesics invariant). In vacuum, the same measurements will yield corresponding components of the Riemann curvature tensor, since the Riemann and Weyl tensors coincide wherever Einstein's vacuum field equations hold.

In a previous paper [1] a physical interpretation was given to the Riemann tensor and a method of measuring its components by observations of test particles was described. The whole discussion there was explicitly metrical, and required the use of clocks (or rigid rods) as well as light rays. It was understood that a particular Riemannian space-time was given in advance, and the arguments depended on the measurement of proper times.

In the present paper, on the other hand, the arguments are entirely *conformal-invariant*; both geometrical and physical interpretations refer not to a particular Riemannian space-time, but to a whole class of space-times which may be obtained from one another by conformal transformations of the metric. Such a class of Ri mannian space-times is called a *conformal space-time*; thus a conformal space-time $C_{(4)}$ is a (sufficiently) differentiable manifold endowed at each of its points P with a real infinitesimal *null cone**)

$$g_{ab}\, dx^a\, dx^b = 0. \tag{1}$$

*) Latin indices $a, b, c, \ldots$ range and sum over 1, 2, 3, 4.

Physically, the null cone is the history of a wave front of light collapsing to and emitted from the event P. The quadratic form (1) must have hyperbolic normal signature in order that the proper distinctions between past and future may be preserved.

The null cone (1) determines the metric $e^{2\sigma} g_{ab}$ in any of the Riemannian space-times of the conformal class up to a *gauge factor* $e^{2\sigma}$ which is an arbitrary function of position. A particular Riemannian space-time may be selected by assigning the gauge. It is often easier to carry out proofs of theorems in a particular gauge, exhibiting their conformal-invariance afterwards, than to devise a strictly conformal-invariant proof.

It is evident that *the ratio of the magnitudes of two vectors or of two simple bivectors at the same point, and the angle between two directions at the same point (especially, the orthogonality of two directions at the same point)* are well-defined in a conformal space-time. It is in fact easy to show how these quantities may be determined by experiments with light signals alone.

In par. 2 we give conformal-invariant definitions of null geodesics and of preferred parameters on them. In par. 3 we give a conformal-invariant definition of infinitesimal shear, and state our main result, which connects the second parameter-derivative of the shear with the conformal curvature tensor. The infinitesimal shear was introduced by Sachs [2], [3] in his analysis of null geodesic congruences, and many of the ideas employed here were developed originally, in metrical form, by him.

2. Null geodesics and preferred parameters

A *null hypersurface* is defined, conformal-invariantly, as a hypersurface which is tangent at each of its points to the infinitesimal null cone at that point. Equivalently, a null hypersurface contains at each of its points exactly *one* null direction. A *null geodesic* is a null curve which lies entirely in a null hypersurface (this manifestly conformal-invariant definition reduces to the usual one as soon as a gauge is assigned). Physically, a null geodesic is the world-line of a light ray, that is, the history of a light pulse.

We show now how to define a conformal-invariant preferred parameter u along any null geodesic of a given congruence. Let l be the selected geodesic and Δx^a its infinitesimal tangent vector at any point P (Fig. 1). Let l_1, l_2, and l_3 be three neighbouring null geodesics, l_1 and l_2 being chosen so that the connecting vectors $\delta_1 x^a$, $\delta_2 x^a$ from P to l_1, l_2 are orthogonal to Δx^a, and l_3 being chosen so that the connecting vector $\delta_3 x^a$ from l to l_3 is *not* orthogonal to Δx^a. Then a parameter u may be defined along l by the condition.

$$(2) \qquad \frac{\text{area of parallelogram spanned by } \Delta x^a/\Delta u \text{ and } \delta_3 x^a}{\text{area of parallelogram spanned by } \delta_1 x^a \text{ and } \delta_2 x^a} = \text{constant along } l.$$

It can be shown that this definition determines the parameter u up to a linear transformation with coefficients constant along l, irrespective of the choice of the neighbouring geodesics l_1, l_2, l_3 and of the choice of connecting vectors from P.

If the gauge is chosen so that the denominator of (2) is constant along l (which corresponds to a zero magnification rate along l in the terms of Sach's analysis),

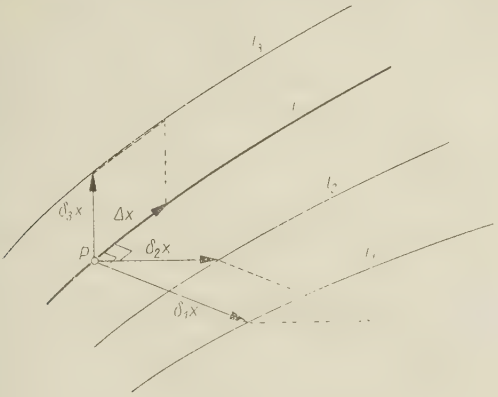

Fig. 1

then the parameter u may be identified with the usual preferred parameter along a null geodesic in the corresponding Riemannian space (cf. [4]).

3. Infinitesimal shear and its propagation

We now define in a conformal-invariant way the *infinitesimal shear* for a congruence of null geodesics (cf. [2]). Let S_P and S_Q be infinitesimal 2-elements orthogonal to a null geodesic l at neighbouring points P and Q of l, and let C be an infinitesimal circle with centre P, lying in S_P (Fig. 2). Those null geodesics of the congruence which meet S_P in the circle C will meet S_Q in an ellipse E. The *infinitesimal shear* of the congruence from P to Q, $d\varepsilon$, is defined by the equation

$$\frac{\text{length of major axis of } E}{\text{length of minor axis of } E} = 1+2d\varepsilon. \tag{3}$$

It may be shown that both $d\varepsilon$ and the major axis e^a of the ellipse E are determined uniquely by the congruence of null geodesics and the points P and Q, and that $d\varepsilon$ is independent of the choice of the orthogonal 2-elements S_P and S_Q.

The physical interpretation of this construction has been given, for the metrical case, by Sachs [2]. Suppose that light passes normally through a small flat transparent circular disc and falls normally on a screen nearby. At a certain instant (P), the disc becomes momentarily opaque, and throws a shadow on the screen (Q). Refraction of the light by the gravitational field makes the shadow elliptical. The null geodesics represent the light rays: the circle C represents the periphery of the disc and the ellipse E the periphery of the shadow. Sachs has shown that the shape, size and orientation of the shadow depend only on the choice of P and Q and not on the velocities of the disc or the screen. It is evident from the above construction that $d\varepsilon$

is conformal-invariant. It can be shown that the *orientation* of the shadow also is conformal-invariant; clearly, the size is not.

We can now state our main result, which relates the propagation of the shear along a null geodesic to the conformal tensor. It is that

$$\frac{d^2\varepsilon}{du^2} = C^a_{bcd}\frac{dx^b}{du}\frac{'dx^c}{du}Pa^b. \tag{4}$$

Here C^a_{bcd} is Weyl's conformal curvature tensor: in any Riemannian space of the conformal class,

$$C^a_{bcd} = R^a_{bcd} + g_b\,[_d R^a{}_c] + R_b\,[_d\delta^a{}_c] - \frac{1}{3}g_b\,[_d\delta^a{}_c]\,R,$$

R^a_{bcd} is the Riemann curvature tensor, defined for example by the commutation rule for covariant differentiation, $V_{b;\,cd} - V_{b;\,dc} = R^a_{bcd}\,V_a$, for any vector V_a, $R_{bc} = R^a_{bca}$ is the Ricci tensor, and $R = g^{bc}R_{bc}$. Also dx^b/du is tangent vector to the geodesic, and $P_a{}^b$ is the projection operator in the direction of the major axis e^a of shear: $P_a^b\,e^a = e^b$, $P_a^b\,f^a = 0$ for every vector f^a orthogonal to e^a.

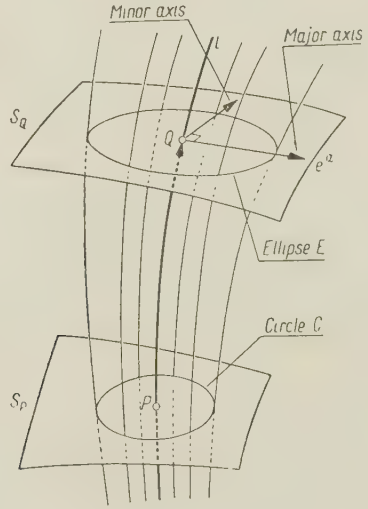

Fig. 3

Eq. (4) determines a physical component of the tensor C^a_{bcd}. Since C^a_{bcd} is irreducible under transformations of the local orthogonal frame at any point (i.e. under Lorentz transformations), all the components of C^a_{bcd} may be determined by examining enough congruences of null geodesics, that is, by carrying out appropriate experiments with light rays.

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Quantum Phenomena in Optical Images

by

R. S. INGARDEN

Presented by W. RUBINOWICZ on May 22, 1961

From the point of view of quantum physics the existing theory of diffraction and, therefore, also all the theory of optical instruments is as if in the air, devoid of solid theoretical foundations. This is true not only with respect to approximative theoretical methods, but also to the so-called rigorous diffraction theory. All up-to-date theories of non-spectroscopic optics (cf., e.g., [1], [2] or any hand-book articles on optics) are essentially classical (i.e. non-*quantum*) and as such are based directly on experiment disregarding quantum phenomena. That the latter cannot be always neglected in this field was shown experimentally in a very striking manner, e.g., by Rose [3]. In general, quantum phenomena are quite sensible at weak illuminations, as e.g. in astronomy, at strong magnifications in microscopy, etc. But from the theoretical point of view also at high illuminations the situation is not satisfactory. The laws of great numbers alone of the probability theory are insufficient to fill the gap between quantum electrodynamics and classical optics. Indeed, averaging over quantum fluctuations is not so simply possible as is generally assumed by practical opticians, since photons cannot be localized in space (cf., e.g., [4]). This is connected with the impossibility of ordinary probabilistic interpretation in theories based on equations with the second time derivative. Further difficulty is that diffraction theory works essentially with monochromatic (or nearly monochromatic) light and with unmoving and infinitely sharp determined space obstacles (stops, etc.). Both notions are non-relativistic (a distinguished frame of reference) and the latter also in contradiction with the mentioned non-localizability of photons. Consequently quantum electrodynamics and nonspectroscopic optics developed for more than half a century side-by-side without mutual influence, independently contacting with experiment, a fact rather paradoxical. A change in this state of affairs is desirable not only from the purely methodological view (in order to *σοζειν τὰ φαινόμενα*, as the ancients said), but also for practical reasons. Recently, namely, the information theory applied to optical instruments made bridging of the gap between both fields necessary and inevitable, cf., e.g., [5]. The present paper would be a further step in this direction.

It is well known that for one photon it is possible to introduce the wave function $\varphi(\boldsymbol{k}, t)$ in the 3-dimensional momentum (wave vectors) space $\boldsymbol{k} = (k_x, \mathrm{m}\, k_y, k_z)$ satisfying the equation

$$i\hbar \frac{\partial \varphi(\boldsymbol{k}, t)}{\partial t} = c\hbar k \varphi(\boldsymbol{k}, t), \tag{1}$$

where $k = |\boldsymbol{k}|$, c is the light velocity, $\hbar$ — Planck's constant and $i = \sqrt{-1}$, (cf. [6] and also [7] or [4]). In the position space $\boldsymbol{x} = (x, y, z)$ we get $\psi(\boldsymbol{x}, t)$ and

$$i\hbar \frac{\partial \psi(\boldsymbol{x}, t)}{\partial t} = c\hbar \sqrt{-\Delta}\, \psi(\boldsymbol{x}, t), \tag{2}$$

where $\sqrt{-\Delta}$ is a non-local (integral) operator defined by its action on Fourier components of any function

$$\sqrt{-\Delta}\, e^{i\boldsymbol{k}\boldsymbol{x}} = k e^{i\boldsymbol{k}\boldsymbol{x}}, \tag{3}$$

where Δ is the Laplacian operator, $(\sqrt{-\Delta})^2 = -\Delta$*).

Now we introduce *a complementary condition* for the wave function corresponding phenomenologically to the existence of a monochromatic source of light

$$(k^2 - k_0^2)\, \varphi(\boldsymbol{k}, t) = 0 \quad \text{or} \quad (\Delta + k_0^2)\, \psi(\boldsymbol{x}, t) = 0. \tag{4}$$

Since (4) is satisfied by $\varphi(\boldsymbol{k}, t) = \delta(k^2 - k_0^2)\, \chi(\boldsymbol{k}, t)$ with arbitrary $\chi(\boldsymbol{k}, t)$ ($\delta(x)$ — the Dirac delta function), it follows from (4)

$$\left(\frac{1}{k} - \frac{1}{k_0}\right) \varphi(\boldsymbol{k}, t) = 0. \tag{5}$$

Therefore, Eq. (2) may be written in the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{c\hbar}{k_0} \Delta \psi(\boldsymbol{x}, t). \tag{6}$$

The non-relativistic character of (6) follows from such a character of the complementary condition (4). Eq. (6) is identical with the Schrödinger equation for a non-relativistic free particle

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2\mu} \Delta \psi(\boldsymbol{x}, t), \tag{7}$$

if we put for the "effective mass" of the photon

$$\mu = \frac{\hbar k_0}{2c} = \frac{\pi \hbar}{c\lambda_0} = \frac{\hbar \omega_0}{2c^2}. \tag{8}$$

*) It must be remarked that φ and ψ are generally considered as vector functions, we apply here, however, the Green-Wolf point of view [8], [9] considering these functions as complex scalars. Moreover, we note that it is generally stated, cf., e.g. [4] § 2.2, that linear dimensions of domains of non-localizability of ψ are of the order of the wave lenght λ. In general, it is not true, since in (3) it must be intergrated over all wave numbers, i.e. over all possible λ's. If only one $\lambda = \lambda_0$ is possible (monochromatic light, cf. below), no integration is necessary and there is no non-localizability.

We see that this mass is equal to the "relativistic mass" of photons with frequency ω_0 in zeroth oscillations, i.e. to their energy expressed in mass units. From Table I it is seen that μ is, in general, extremely small in comparison with the electron mass m_0.

TABLE I

Spectral region	λ_0 (cm.)	$\frac{\mu}{m_0}$
Radio waves	10^4	10^{-15}
Microwaves	10	10^{-11}
Visible light	10^{-5}	10^{-6}
Hard X-rays	10^{-9}	10^{-1}

Putting

$$\psi(\boldsymbol{x}, t) = u(\boldsymbol{x})\, e^{-ik_0 ct} \tag{9}$$

we get from (6)

$$(\Delta + k_0^2)\, u(\boldsymbol{x}) = 0, \tag{10}$$

i.e. the Helmholtz equation which is the base of classical diffraction theory. For solutions of the form (9) no dispersion of waves is possible in spite of the type of (6), i.e. no group velocity as for electrons. (The phase velocity is equal to c). We must exclude, however, all other stationary solutions of (6) which have no optical interpretation. Therefore, we introduce *the second complementary condition*

$$E = k_0 \hbar c = \hbar \omega_0. \tag{11}$$

Because of the infinite degeneracy of the energy level (11), the Hilbert space of our quantum problem (6) with (11) has yet an infinite number of dimensions, as usual in quantum mechanics. Now we may apply without any modification the ordinary non-relativistic method of second quantization for bosons, (cf. e.g. [10] § 62) and we get full quantum theory of diffraction. Also the usual non-relativistic probability interpretation of $|\psi|^2$ is possible without restriction (the non-localizability of photons being liquidated by (4)). It seems, therefore, that in such a way the classical diffraction theory together with its ordinary interpretation is "saved" from the point of view of quantum physics (and its conformity with experiment explained). Simultaneously, however, new possibilities of a further development of this theory in the pure quantum direction are opened.

It must be pointed out that in diffraction problems interesting for the theory of optical instruments, the Hilbert space of one photon is reduced to that with a finite number of dimensions. This is affected by the presence of two stops: the entrance (or exit) pupil and the field stop. If the instruments have rotational symmetry, we obtain, namely, in the Gaussian domain (for sufficiently small aperture and field angles), using cylindrical co-ordinates (cf. [5]),

$$u(r, \varphi, z) = \sum_{l=1}^{l_0} \sum_{m=1}^{l} i^m A_{lm} B_{lm}(r, z)\, e^{im\varphi}, \tag{12}$$

where

$$B_{lm}(r, z) = \int_0^{\varrho_0} J_m(\varrho, r)\, J_0\left(a_l \frac{\varrho}{\varrho_0}\right) e^{iz\sqrt{\varrho_0^2 - \varrho^2}}\, \varrho\, d\varrho \tag{13}$$

(a_l — zeros of the Bessel function J_0) and

$$l_0 = \frac{2}{\pi\lambda_0}\sqrt{SA}, \qquad \varrho_0 = \frac{2}{\lambda_0}\sqrt{\pi A}. \tag{14}$$

Here A denotes the angular aperture of the instrument (in the solid angle measure), S = the area of image. The number of independent complex parameters A_{lm}, i.e. the number of complex degrees of freedom is in this case

$$N = l_0^2 = \frac{4}{\pi^2}\frac{SA}{\lambda_0^2}, \tag{15}$$

in accordance with the result first derived by Laue [11] and then independently by Gabor [13], (cf. also [12] and [5])*).

If we consider the given intensity of light, we have a definite number of photons M_0 (full energy $W = M_0 E = M_0 \hbar\omega_0$). The number M_0 is, however, not identical with the number M_1 of photons registered by means of any detecting device (e.g. a photographic layer): $M_1 \leqslant M_0$, but, in general $M_1 \ll M_0$. M_1 depends not only on M_0 but also on properties of the sensitive layer, in the first approximation on its "effective charge", cf. [5], i.e. the "test charge" which intervenes in the Bohr-Rosenfeld uncertainty relations.

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*) Dr. A. Lohmann (Braunschweig) has kindly drawn the attention of the author to Laue's article [12]. Laue has considered, in particular, the case of the rectangular symmetry, where the dimensionless coefficient is equal to 1.

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Dipole Array of Ferroelectrically Active *A*-Ions in ABO_3-Substances

by

A. JAŚKIEWICZ and H. KONWENT

Presented by W. RUBINOWICZ on May 25, 1961

The present authors have given a method of determining ferroelectric and antiferroelectric structures in perovskite-type crystals which are formed when the crystal transforms to the polar state [1]. In that paper we discussed an arrangement of ionic dipoles formed by a displacement of a *B*-ion in the ABO_3-substance. Here we shall discuss the dipole arrangement in the perovskite-type substances in the polar state when the *A*-ion is ferroelectrically active.

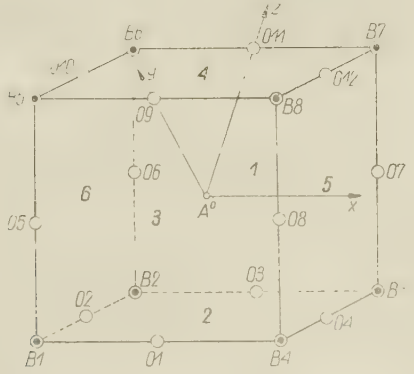

Fig. 1. Elementary cell of perovskite-type crystal, body centered by *A*-ion

Venevtsev and Zhdanov [2] distinguished substances with various moving ions. In [2] it was assumed that in the ABO_3-substances there exist polar states differing by a shift of either an *A*- or *B*-ion. In analogy to the *B*-ion type crystals we assume that the *A*-ion when free to move is displaced towards one of the surrounding oxygen ions. As the *A*-ion has twelve nearest oxygen ions there are twelve asymmetric positions of it.

Considering the *A*-ion as ferroelectrically active, it is convenient to choose the elementary cell (with the lattice constant a) and the co-ordinate system as in Fig. 1.

It is assumed that the A-ion is displaced along the z-axis in the positive direction creating an electric dipole with the moment $\boldsymbol{m}(0, 0, m)$. In the following all dipoles will be treated as point dipoles. For the electric field from this dipole we use the well known formula

$$\boldsymbol{E} = \frac{3\boldsymbol{r}(\boldsymbol{rm}) - r^2\boldsymbol{m}}{r^5}, \tag{1}$$

where $\boldsymbol{r}$ is the radius vector.

Using this formula we calculate the field acting on the B- and O-ions in the primary cell. Denoting the polarizabilities of B- and O-ions by α_B and α_0, respectively, we obtain elastic dipoles induced in the B- and the O-ions by the primary dipole.

The field produced by the primary dipole and the elastically polarized ions of the primary cell acts on the nearest A-ions tending to displace them. The polarized cell is surrounded by the six adjoining cells with common faces. Polarization of each common face is already known, the latter being the part of the polarized cell. The adjacent cells get the same number as the common face with the primary cell. We denote the faces of the primary cell by Arabic numbers (see Fig. 1). The A-ion in turn gets the same number as the cell which is body-centered by it.

On the site of an A-ion in a neighbouring cell there acts the electric field produced by the primary ionic dipole and the elastic dipoles induced in the adjoining face. E.g., on the site of the $A1$-ion (the ion that body-centers the cell adjacent to face 1, see Fig. 1) there acts the field produced by the elastic dipoles induced in the ions $B2$, $B3$, $B6$, $B7$, 03, 06, 07, 011 and by the primary dipole AO. Knowing the magnitude of the elastic dipoles we can calculate the components of the total electric field acting on ion $A1$:

$$\begin{cases} E_x(A1) = 0, \\ E_y(A1) = \dfrac{256}{27}\alpha_B\dfrac{m}{a^6} + 27\,\alpha_0\dfrac{m}{a^6} - \dfrac{3}{2}\dfrac{m}{a^3}, \\ E_z(A1) = -\dfrac{256}{27}\alpha_B\dfrac{m}{a^6} - 39\,\alpha_0\dfrac{m}{a^6} + \dfrac{1}{2}\dfrac{m}{a^3}. \end{cases} \tag{2}$$

Computing the total electric field acting at the sites $A2$, $A3$ and $A4$ we get the following components:

$$\begin{cases} E_x(A2) = E_x(A3) = E_x(A4) = E_x(A1), \\ E_y(A2) = -E_y(A3) = E_y(A4) = -E_y(A1), \\ E_z(A2) = E_z(A3) = E_z(A4) = E_z(A1). \end{cases} \tag{3}$$

For the total electric field acting at the sites of $A5$ and $A6$ ions, the components are given below:

$$\begin{cases} E_x(A5) = E_x(A6) = 0, \\ E_y(A5) = E_y(A6) = 0, \\ E_z(A5) = E_z(A6) = -\dfrac{m}{a^3}\left(14\,\dfrac{\alpha_0}{a^3} + 1\right). \end{cases} \tag{4}$$

The only non-vanishing component of the electric field acting on the *A*5- and *A*6-ions is the *z*-component. This component is opposite with respect to direction of the primary ion tending, therefore, to displace the *A*5- and *A*6-ion in the negative direction. From formulae (4) we see that at least two nearest neighbour *A*-ions are antiparallel shifted. Thus, we see that in the case discussed only the antiferroelectric dipole arrangement may be formed. We can discuss, using formulae (3) and (4) the, possible antiferroelectric arrangements that will develop in the considered crystal class.

The electric field given by formula (3) has two non-vanishing components, the *x*-component being zero. Since we are interested in the shift of the *A*-ion towards one of the neighbouring oxygen ions, we will discuss the dipole arrangement which will develop in such a case. The existence of the *y*- and *z*-components of the field

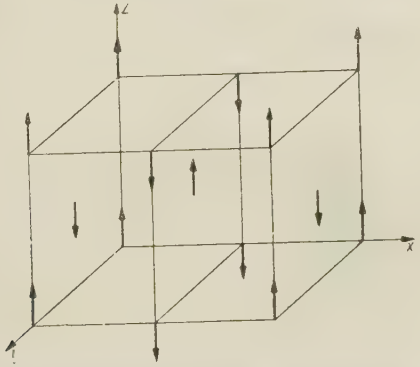

Fig. 2. Arrangement of dipoles in the lattice formed by displacement of *A*-ions in the *z*-direction

acting on the *A*-ion make it possible that the *A*-ion is shifted either along *y*-axis or *z*-axis. We will discuss, therefore, two possible antiferroelectric arrangements, if: 1) the *A*-ion is shifted along *z*-axis, 2) the *A*-ion is shifted along *y*-axis.

In the first case, the *A*-ions are displaced along the *z*-axis, the orientation of dipoles may be inferred from the electric field acting on the neighbouring ions of the primary ion and given by formulae (3) and (4). This dipole array is presented in Fig. 2.

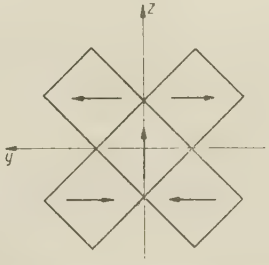

Fig. 3. Arrangement of dipoles formed by displacement of *A*-ions in the *y*-direction in the nearest environment of the primary *A*-ion

In the second case, the A-ions form dipoles at a right angle to the direction of the primary dipole, the array of dipoles, as inferred from formulae (3) and (4), is presented in Fig. 3. The total dipole arrangement that will develop in the crystal is pictured in Fig. 4.

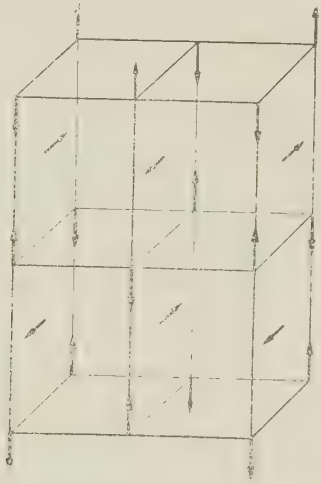

Fig. 4. Spatial arrangement of A-dipoles when A-ions in the nearest environment of the primary A-ion are displaced along the y-direction

As the above considerations indicate, in the perovskite-type crystals, there are formed only antiferroelectric dipole arrangements when an A-ion is displaced towards one of the surrounding oxygen ions.

We are greatly indebted to Professors R. S. Ingarden and J. Nikliborc for their valuable suggestions and discussion.

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Vibrational Excitation of Diatomic Molecules Due to Nuclear Recoil

by

J. FIUTAK and L. WOLNIEWICZ

Presented by A. JABŁOŃSKI on May 15, 1961

It is well known, that in the Born-Oppenheimer approximation the problem of finding the probability of an excitation of a molecule can be divided into two separate problems: the probability of the electronic excitation, and the probability of the vibrational and rotational excitation. We shall discuss only the second question. In the case of an excitation due to nuclear recoil, it has been solved by Steinwedel and Jensen [2] and Wolfsberg [3] in the harmonic oscilator approximation. The aim of this paper is to present a method allowing to perform similar calculations in the approximation of the Morse potential. The numerical computations have been carried out for the H_2 molecule, for which the effect of anharmonicity is expected to be comparatively important.

Let us denote by M_A, M_B, $\vec{R}_A$, $\vec{R}_B$ the masses and co-ordinates of the nuclei. Now, if we assume that the nucleus A suffers a recoil $\vec{t}$, the probability of the excitation in which we are interested in reads:

$$P_{vl\to v'l'} = |\langle v', l'| \exp(ikR\cos\theta)| v, l\rangle|^2, \tag{1}$$

where

$$\vec{k} = \frac{M_B}{M_A+M_B}\vec{t}; \quad \vec{R} = \vec{R}_A - \vec{R}_B; \quad \cos\theta = \frac{\vec{k}\cdot\vec{R}}{kR},$$

and v, l, v', l', are the vibrational and rotational quantum numbers of the initial and final states, respectively.

At the beginning we shall discuss those relations, which follow from (1) and are independent of the form of the vibrational potential. For the mean energy-transfer $\varepsilon(k)$ one has obviously

$$\varepsilon(k) = \sum_{v'l'} (E_{v'l'} - E_{vl})\, P_{vl\to v'l'}. \tag{2}$$

where E_{vl} is the energy of a state specified by v, l. By making use of the identity

$$[\exp(-i\vec{k}\vec{R}), [H, \exp(i\vec{k}\vec{R})]] = \frac{\hbar^2 k^2}{2\mu}, \tag{3}$$

$$H = -\frac{\hbar^2}{2\mu}\Delta_{\vec{R}} + V; \qquad \mu = \frac{M_A M_B}{M_A + M_B}$$

one gets easily $\varepsilon = \frac{\hbar^2 k^2}{2\mu}$, and (2) reads now

$$\frac{\hbar^2 k^2}{2\mu} = \varepsilon(k) = \sum_{v' l'} (E_{v' l'} - E_{vl})\, P_{vl \to v' l'}. \tag{4}$$

Since we are interested in the effect of anharmonicity only, we shall neglect the coupling of the vibrational and rotational motions. So we can write

$$E_{vl} = E_v + E_l; \qquad |vl> = |v>|l>;$$

$$(H_R - E_v)|v> = 0; \qquad H_R = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial R^2} + V.$$

Now it follows that

$$[\exp(-i\vec{k}\vec{R}), [H_R, \exp(i\vec{k}\vec{R})]] = \varepsilon(k)\cos^2\theta,$$

and one gets for the mean vibrational energy $\varepsilon^v(k)$:

$$\varepsilon^v(k) = \sum_{l' v'} (E_{v'} - E_v)\, P_{vl \to v' l'} = \varepsilon(k) <|\, l \cos^2\theta |\, l>. \tag{5}$$

In particular, for a molecule being initially in the ground state $v = 0$, $l = 0$ one has $\varepsilon^v = \frac{1}{3}\varepsilon$.

So in the approximation used, ε and ε^v are independent of the vibrational potential.

Let us investigate the effect of anharmonicity on the mean value v' of the quantum number v'. By making use of (5) one gets for the harmonic (h) and anharmonic (a) potential, respectively:

$$\bar{v}'_h = \sum_{v'} v' \sum_{l'} P^h_{00 \to v' l'} = \frac{1}{3}\eta(k), \tag{6}$$

$$\bar{v}'_a = \sum_{v'} v' \sum_{l'} P^a_{00 \to v' l'} = \frac{1}{3}\eta(k) + \frac{\omega_e x_e}{\omega_e}\overline{v'(v'+1)}_a, \tag{7}$$

where $\eta(k)$ is the mean energy-transfer in unites of $\hbar\omega_e$. In the derivation of (7) we have assumed:

$$E^a_v = \hbar\omega_e\left(v + \frac{1}{2}\right) - \hbar\omega_e x_e\left(v + \frac{1}{2}\right)^2,$$

The last term in the righthand side of (7) is a small correction. So in the first approximation we replace $\overline{v'(v'+1)_a}$ by $\overline{v'(v'+1)_h}$. The latter mean value can be found by the method given in [3]. After performing this one obtains

$$\bar{v}'_a = \frac{1}{3}\left(1+2\frac{\omega_e x_e}{\omega_e}\right)\eta + \frac{\omega_e x_e}{5\omega_e}\eta^2. \tag{8}$$

It is seen, that $\bar{v}'_a > \bar{v}'_h$. We have computed the mean values for the H_2 molecule. The numerical results for various η are presented in Table I.

TABLE I

η	$\bar{v}'_a$	$\bar{v}'_h$	$\frac{\bar{v}_a - \bar{v}_h}{\bar{v}_h}$ in %
1	0.357	0.333	7.2
2	0.724	0.667	8.5
3	1.102	1	10.2
4	1.491	1.333	12
6	2.301	2	15
8	3.153	2.667	18
10	4.049	3.333	21
15	6.476	5	30

TABLE II

η	$P^a_{00\to 0}$	$P^h_{00\to 0}$	$\frac{P^h - P^a}{P^h}$ in %
0.6	0.826	0.831	0.7
0.8	0.780	0.787	0.9
1.0	0.739	0.747	1.1
1.2	0.701	0.711	1.4
1.4	0.668	0.678	1.5
1.6	0.636	0.649	2.0
1.8	0.607	0.622	2.4
2.0	0.579	0.598	3.2

In further investigation we shall restrict ourselves to the Morse potential. In this case one has:

$$P^a_{00\to v} = \sum_l P^a_{00\to vl} = \sum_l |\langle v|\langle l|\exp(ikR\cos\theta)\,|\,0\rangle\,|0\rangle|^2 = \tag{9}$$

$$= \langle l=0|\,F_{0v}(\theta)|\,l=0\rangle,$$

where

$$F_{0v}(\theta) = |\langle v\,|\exp(ikR\cos\theta)|\,0\rangle|^2 = |\langle v|\exp[ik(R-R_0)\cos\theta]|\,0\rangle|^2, \tag{10}$$

$$R_0 = \langle 0|R|0\rangle.$$

In (10) the brackets denote the integration with respect to R only.

By expanding (10) in a power series, one expresses $P_{00\to v}$ in terms of the integrals $\langle v|(R-R_0)^n|\,0\rangle$, which can be evaluated by the method given in the appendix. Obviously, this procedure is applicable to the evaluation of any $P_{vl\to v'l'}$. It is however convenient only in the case of $\eta \sim 1$.

The results of numerical calculations of the probability $P^a_{00\to 0}$ for the H_2 molecule are listed in Table II. Corresponding results for the harmonic potential are also given for comparison.

From the numerical results one sees, that for large η, both the mean value v' of the quantum number v' and the probability of the vibrational excitation, $1 - P_{00\to 0}$ increase appreciably in the anharmonic case. As a consequence of $E^a_v < E^h_v$, those results are in full agreement with our general statement (5), that the mean energy ε^v is independent of the form of the potential.

We would like to express our gratitude to Dr W. Kołos for reading and commenting on this paper.

Appendix

The integrals $\langle v|(R-R_0)^n|v'\rangle$ have been evaluated as follows: By making use of

$$[H_R, R] = \frac{\hbar}{i}\frac{p}{\mu}; \qquad [H_R, p] = -\frac{\hbar}{i}\frac{dV}{dR},$$

one gets in the case of $v' \neq v$.

$$\langle v|R-R_0|v'\rangle = -\frac{1}{\mu}\frac{\left\langle v\left|\frac{dV}{dR}\right|v'\right\rangle}{(E_v-E_{v'})^2}. \tag{11}$$

For the Morse potential $V = D\{1-\exp(-\alpha R+\alpha R_e)\}^2$, (11) reads

$$\langle v|R-R_0|v'\rangle = -\frac{2D\alpha}{\mu K}\frac{\left\langle v\left|z-\frac{1}{K}z^2\right|v'\right\rangle}{(E_v-E_{v'})^2}, \tag{12}$$

where

$$z = K\exp[-\alpha(R-R_e)]; \qquad K = \frac{\omega_e}{\omega_e x_e}.$$

On the other hand, in the case $v' = v$ one has:

$$\langle v|R-R_e|v\rangle = -\frac{1}{\alpha}\left\langle v\left|\lg\frac{z}{K}\right|v\right\rangle. \tag{13}$$

The right-hand sides of (12) and (13) can be evaluated by making use of the integral representation of $|v\rangle$ given by Pekeris [1]. One gets finally

$$v' > v \quad \sqrt{2\mu\omega^e}\langle v|R-R_0|v'\rangle =$$

$$\frac{(-1)^{v'-v}}{v-v'}\left\{K^{v'}\frac{v'!}{v!}\left(1-\frac{(v-v')^2}{(K-1-v-v')^2}\right)\frac{\Gamma(K-v)}{\Gamma(K-v')}\right\}^{1/2},$$

$$v' = v \quad \sqrt{2\mu\omega_e}\langle v|R-R_0|v'\rangle =$$

$$-K^{1/2}\left\{\sum_{j=1}^{2v}\frac{1}{K-1-j}+v!\sum_{s=1}^{v}\frac{(-1)^{s+1}}{s(v-s)!}\frac{\Gamma(K-2v)}{\Gamma(K-2v+s)}\right\},$$

and obviously

$$\langle v|(R-R_0)^n|v'\rangle = \sum_{v''}\langle v|(R-R_0)^{n-1}|v''\rangle\langle v''|R-R_0|v'\rangle.$$

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EXPERIMENTAL PHYSICS

Anomalous Phosphorescence of Naphthalene in Methyl Methacrylate Polymer

by

S. CZARNECKI

Presented by W. JABŁOŃSKI on May 25, 1961

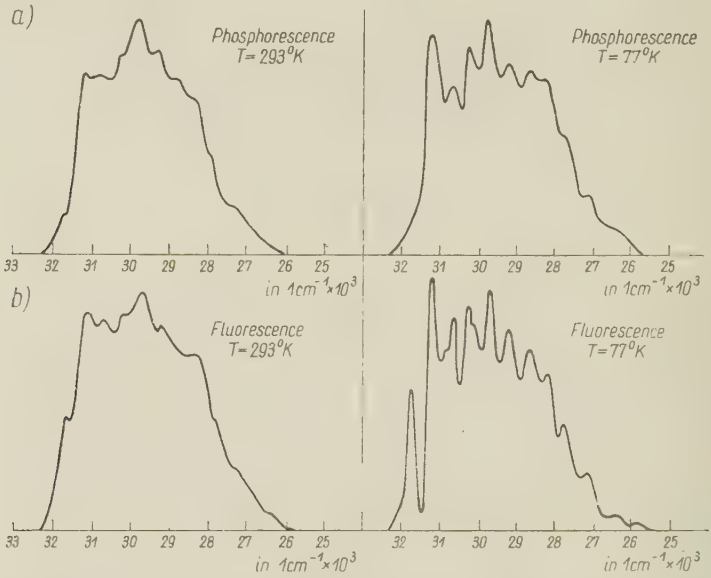
It is well known, that a great many organic compounds in the solid solutions show an afterglow besides fluorescence. Two kinds of afterglow are usually observed: one has an identical spectrum with that of fluorescence (phosphorescence, or α-band) and the other has its spectrum shifted towards the longer wave-lengths (fluorescence of long duration, or β-band). Both kinds of afterglow can be explained by the molecular energy level diagram proposed by A. Jabłoński [1], who assumes a metastable level M between the fluorescence F level and the ground N level.

In a strongly excited phosphor a rather large concentration (up to 80%) [2] of molecules on the M level can be obtained, the transitions between F and M levels being almost exclusively of the radiationless type. The spontaneous $M \rightarrow N$ transitions, though strongly forbidden, result in fluorescence of long duration (β-band). The condition that the phosphorescence (α-band) appears is that the molecules are activated from the M to the F state. Such an activation is brought about by the thermal energy of the system and therefore the phosphorescence can appear only above a certain temperature, characteristic for a given molecule. This temperature depends on the energy difference between M and F levels (known as temperature of activation).

In this work the afterglow of naphthalene, α-bromonaphthalene, α-iodonaphthalene and phenanthrene in rigid solutions of the methyl methacrylate polymer has been investigated. The known amount of the investigated compound was introduced into the monomer, then the solutions were polymerized by heating without any catalyser. The activator concentrations were $c = 1 \cdot 10^{-3}$ gm./cm.^{3}.

The activation temperatures for the hydrocarbons investigated, given by P. Pringsheim, were comparatively high, up to 100°C [3], so that the phosphorescence should not have appeared even at room temperature. However, the phosphorescence of naphthalene was found not only at room temperature, but also at liquid nitrogen temperature. The intensity of this phosphorescence was weaker by two orders of magnitude than that of the fluorescence of long duration, and practically, did not dependend on temperature.

The Figure shows the densitometer curves of the phosphorescence spectra of naphthalene at 293°K and 77°K (a) compared with those of the fluorescence at the same temperatures (b). The photographs were taken using a two discs phosphoroscope and a small KIPP quarz spectrograph (the slit was 0.1 mm. and ortochromatic Raman Platten — Agfa were used).



The inspection of the spectra shows beyond doubt that their origin is due to phosphorescence. The phosphorescence spectra have their structure somewhat more diffuse. A pronounced decreasing of the shortest wavelength band (31750 cm.$^{-1}$) at room temperature and its disappearing at the temperature of the liquid nitrogen is due to the reabsorption connected with a different position of the sample in regard to the spectrograph and the beam of the exciting light.

The above mentioned phosphorescence appeared only in naphthalene, it could not, however, be detected for the other three compounds despite many hours of exposition.

An interpretation may be attempted here by assuming that energy is transferred among the naphthalene molecules remaining at the metastable level. The metastable level of naphthalene lies comparatively deep under the fluorescence level, close to the half distance between N and F. Assuming that one of the naphthalene molecules in the M state takes over the excitation energy from the other, the latter would undergo disactivation by falling down on the N ground level, and the former would turn up at one of the higher oscillation levels of the fluorescence level F. In the above process the Wigner's rule (the conservation of total spin) is fulfilled. If the biradical theory of the metastable state is assumed [4] then the following scheme can be proposed

$$A^* \uparrow\uparrow + A^* \downarrow\downarrow \rightarrow A^{**} \uparrow\downarrow + A \uparrow\downarrow ,$$

where A^* is the molecule in the M state, and A^{**} — the molecule in the F state. Thus, before the energy is transferred, both molecules are in the triplet state, and afterwards they go over to the singlet states, the one of them being excited to the F level, whereas the other one, disactivated, to the N ground level.

For phenanthrene there has been no phosphorescence found and this effect can easily be explained by means of the above mechanism. The M level of phenanthrene lies much closer to the F level and the "double excited" molecule cannot reach any of the oscillation levels of the F state.

The phosphorescence has not been found also for α-bromonaphthalene and α-iodonaphthalene. For these the energy difference $F - M$ is almost identical with that of the naphthalene. The explanation may be provided if the short lifetime of the M state is considered compared with that of naphthalene and phenanthhrene. The short lifetime of these molecules on the M level should appreciably lower the probability of the energy transfer.

The above probability should be affected by the mean distance between the activator molecules, i.e. by concentration. The valuable information about the mechanism of the effect in the case of the naphthalene can be obtained from investigations of the phosphorescence intensity dependence on the concentration. Such investigations are in progress.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ
И ФИЗИЧЕСКИХ НАУК

1961

ТОМ IX Резюме статей ВЫПУСК 7

В ранее опубликованной в Бюллетене ПАН работе [1] дается определение особого класса решений некоторой параболической системы уравнений, обоснованное на определениях, приведенных в работах [2] и [3].

В настоящей работе проводится подробный математический анализ свойств особого класса решений, называемых особыми квази-решениями.

В работе доказывается, что аксиома Геделя $V = L$ (сравн. [4]) находится в противоречии с аксиомой устанавливающей, что имеется кардинальное число $\mathfrak{m}$ такое, что в множестве мощности $\mathfrak{m}$ имеется бивалентная счетно аддитивная мера.

Автор исследует задачу Гильберта, когда нелинейное условие вида:

$$\Phi_\nu^+(t) = \sum_{\beta=1}^{n} G_{\nu\beta}(t)\,\Phi_\beta^-(t) + g_\nu(t) + F_\nu[t, \Phi_1^+(t), \ldots, \Phi_n^+(t), \Phi_1^-(t), \ldots, \Phi_n^-(t)],$$

$t \in L$, $\nu = 1, 2, \ldots, n$, задается на совокупности L простых замкнутых контуров без общих точек, ограничивающих некоторую конечную связную область на плоскости комплексной переменной. Предполагается, что функции $g_\nu(t)$, $F_\nu(t, u_1, \ldots, u_{2n})$ допускают разрывы, а $G_{\nu\beta}(t)$ — заданные на L функции удовлетворяют условию Гельдера.

Пользуясь топологической теоремой И. Шаудера, автор сформулировал достаточные условия разрешимости задачи.

В работе рассматривается система парциальных уравнений вида

$$(1) \qquad \frac{\partial u_s}{\partial t} = F_s\left(x, t, u_1, \ldots, u_m, \frac{\partial u_s}{\partial x_j}, \frac{\partial^2 u_s}{\partial x_j \partial x_x}\right), \quad x = (x_1, \ldots, x_n),$$

где правые стороны, определенные для точки (x, t), принадлежащей к области D, пространственно неограниченной, и для произвольных значений остальных аргументов.

В Теореме 1 доказывается, что если функции F_s удовлетворяют соответственному условию Липшица с переменными коэффициентами, то первая проблема Фурье для системы (1) обладает лишь одним параболическим решением (в смысле приведенном Я. Шарским) с порядком возрастания не больше функции $\exp K|x|^2$ в области D.

Предметом Теоремы 2 является неравенство вида $\partial u_s/\partial t \leqslant F$, а Теорема 3 касается существования решения первой проблемы Фурье.

Во второй части сообщения доказываются аналогичные теоремы для проблемы с краевыми условиями

$$\frac{du_s}{dl_s} + G_s x, t, u_1, \ldots, u_m = 0 .$$

В работе автор обсуждает симметрию рутила и представления пространственной группы рутила D_{4h}^{14}.

После краткого анализа положения внутри зоны Бриллюэна, приводятся представления нумерированные волновым вектором $\vec{k}$, лежащим на поверхности зоны Бриллюэна. Приняты во внимание все характерные точки, оси и плоскости, а также спин-орбитальное взаимодействие.

В работе дается геометрическая интерпретация тензора Вейля (конформной кривизны) пространства-времени, используя выражения, относящиеся к поведению конгруэнтности нулевых геодезических линий. Соответствующая физическая интерпретация дает, в основном, средства для измерения физических компонент тензора Вейля при помощи лишь лучей света с пренебрежением часов и жестких стержней.

TABLE DES MATIÈRES

Cena zł 20.—